

Zeta functions and decomposition spaces

Andrew Kobin (UC Santa Cruz)

akobin@ucsc.edu andrewkobin.com

preliminary joint work with

Julie Bergner, Matt Feller (UVA)

Classical zeta functions

	K/\mathbb{Q} #field	X/\mathbb{F}_q variety
<u>Riemann</u>	<u>Dedekind</u>	<u>Hasse-Weil</u>
name:	name:	name:

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad | \quad \zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} \quad | \quad Z(X,t) = \exp \left[\sum_{n=1}^{\infty} \frac{\#X(F_q^n)}{n} t^n \right]$$

product formula:

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product formula:

$$\zeta_{\mathbb{Q}}(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$$

$$Z(X,t) = \prod_{x \in |X|} \left(1 - t^{\deg(x)}\right)^{-1}$$

cycle formula:

$$Z(X,t) = \sum_{\alpha} 1 t^{\deg(\alpha)}$$

underlying poset:

underlying poset:

underlying poset:

$$(\mathbb{N}, |)$$

$$(I_K, |)$$

$$(Z_0^{\text{eff}}(X), \leq)$$

arithmetic functions:

arithmetic functions:

arithmetic functions:

$$f: \mathbb{N} \rightarrow \mathbb{C}$$

$$f: I_K \rightarrow \mathbb{C}$$

$$f: Z_0^{\text{eff}}(X) \rightarrow \mathbb{C}$$

Dirichlet
series:

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

convolution:

$$\left(\sum \frac{f(n)}{n^s} \right) \left(\sum \frac{g(n)}{n^s} \right)$$

$$= \sum \frac{h(n)}{n^s}$$

$$h(n) =$$

zeta function:

$$f: n \mapsto 1$$

Möbius function:

$$\mu: n \mapsto$$

Dirichlet
series:

$$\sum_a \frac{f(a)}{N(a)^s}$$

convolution:

$$\left(\sum \frac{f(a)}{N(a)^s} \right) \left(\sum \frac{g(a)}{N(a)^s} \right)$$

$$= \sum \frac{h(a)}{N(a)^s}$$

$$h(a) =$$

zeta function:

$$f: a \mapsto 1$$

Möbius function:

$$\mu: a \mapsto$$

Dirichlet
series:

$$\sum_{\alpha} f(\alpha) t^{\deg(\alpha)}$$

convolution:

$$\left(\sum f(\alpha) t^{\deg(\alpha)} \right) \left(\sum g(\alpha) t^{\deg(\alpha)} \right)$$

$$= \sum h(\alpha) t^{\deg(\alpha)}$$

$$h(\alpha) =$$

zeta function:

$$f: \alpha \mapsto 1$$

Möbius function:

$$\mu: \alpha \mapsto$$

Möbius inversion:

Möbius inversion:

Möbius inversion:

$$\mu * \rho = 1$$

$$\mu * \rho = 1$$

$$\mu * \rho = 1$$



$$\rho(s)^{-1} =$$

$$\rho(s)^{-1} =$$

$$Z(x, t)^{-1} =$$

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$$

$$\sum_a \frac{\mu(a)}{N(a)^s}$$

$$\sum_{\alpha} \mu(\alpha) t^{\deg(\alpha)}$$

Zeta functions for posets

An interval in a (locally finite) poset

(P, \leq) is a sub-poset

$$[x, y] = \{z \in P \mid x \leq z \leq y\}.$$

DEF The incidence coalgebra of a

(locally finite) poset (P, \leq) is the

free k -vector space $C(P)$ on intervals,

with: $\Delta: C(P) \rightarrow C(P) \otimes C(P)$

$$[x, y] \mapsto \sum_{z \in [x, y]} [x, z] \otimes [z, y]$$

DEF The incidence algebra of (P, \leq)

is the dual vector space

$$I(P) = \text{Hom}_k(C(P), k)$$

with multiplication given by convolution:

$$I(P) \otimes I(P) \rightarrow I(P)$$

$$\varphi \otimes \psi \mapsto (\varphi * \psi)([x, y]) :=$$

$$\sum_{z \in [x, y]} \varphi([x, z]) \psi([z, y])$$

In $I(\mathcal{P})$, there are distinguished

elements: $f: \mathcal{C}(\mathcal{P}) \rightarrow k$
 $[x, y] \mapsto 1$

$$\mu: \mathcal{C}(\mathcal{P}) \rightarrow k$$

(recursive formula)

which satisfy $\mu * f = 1 = f * \mu$.

Ex

The Riemann, Dedekind and Hasse-Weil zeta functions, and their Möbius inversion principles, are all special cases.

... need to pass to the reduced

Subtlety: need to pass

incidence algebra, e.g. $I(N) \rightsquigarrow \tilde{I}(N)$
 $[m, n] \rightsquigarrow [1, \frac{n}{m}]$

Fact: One can deduce the product formulas for $S_q(s)$, $S_k(s)$ and $Z(X, t)$ from a decomposition of the corresponding posets N , I_k , $Z_0^{\text{eff}}(X)$.

Decomposition spaces

(due to Gálvez-Carrillo-Kock-Tonks)

Idea: incidence algebras don't just come from posets, but higher homotopy-theoretic structure.

Recall: a simplicial set is a functor

$$X: \Delta^{\text{op}} \longrightarrow \text{Set}$$

(cat. of combinatorial simplices)

$$X_0 \rightleftarrows X_1 \rightleftarrows X_2 \rightleftarrows \dots$$

DEF The incidence coalgebra of a

(locally finite) simplicial set X is

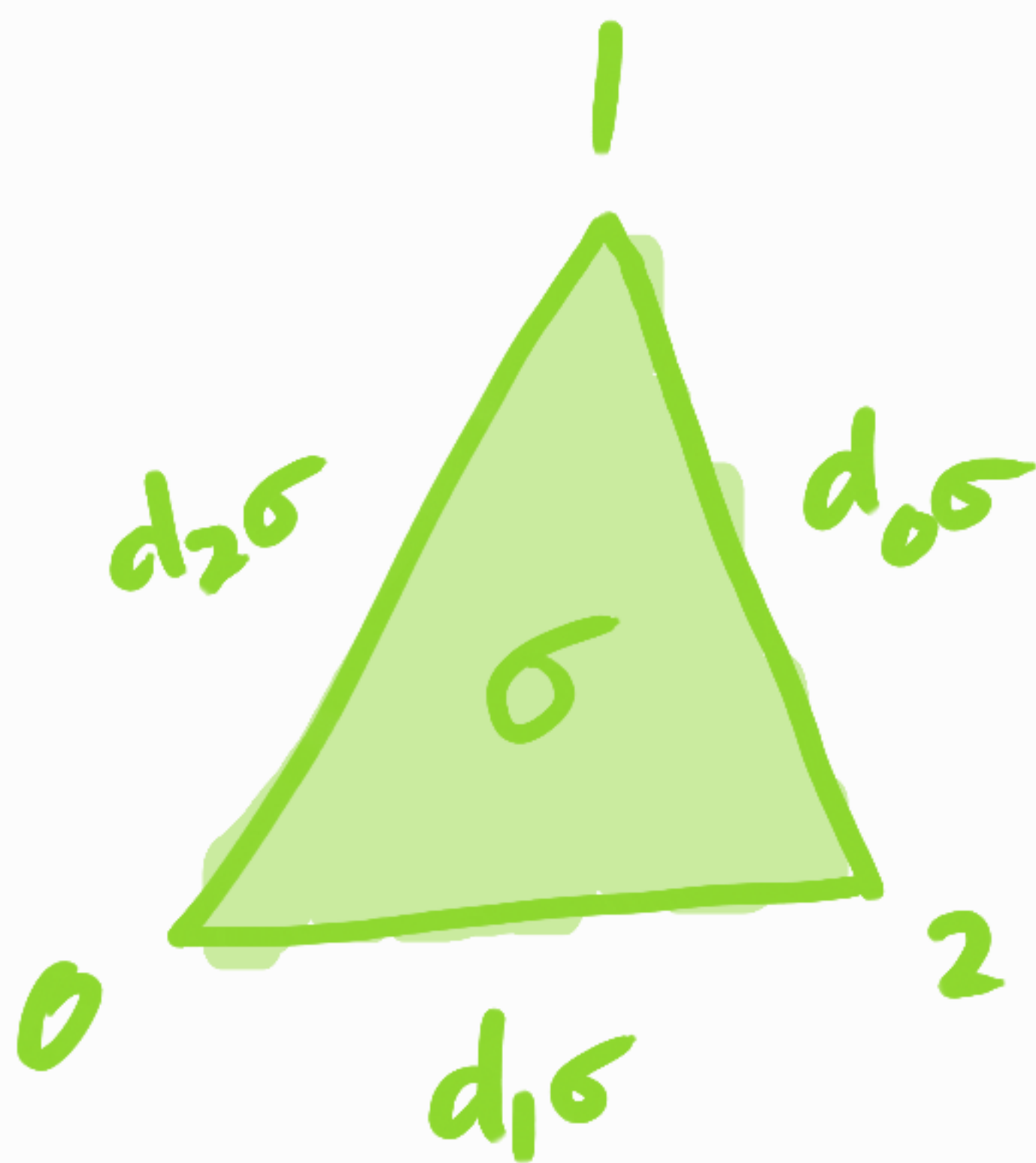
the k -vector space $\mathcal{C}(X) = \bigoplus_{x \in X_i} k$

with $\Delta: \mathcal{C}(X) \rightarrow \mathcal{C}(X) \otimes \mathcal{C}(X)$

$$[x] \mapsto \sum d_2 \sigma \otimes d_0 \sigma.$$

$$\sigma \in X_2$$

$$d_1 \sigma = x$$



Fact / "definition"

$C(X)$ is a

coassociative, counital k -coalgebra

exactly when X is a decomposition

set (aka a 2-Segal set).

Gálvez-Carrillo - Kock - Tonks (and
independently Dyckerhoff - Kapranov)

realize decomposition sets (resp.

generalized decomposition
 2-Segal sets) to decomposition
 spaces (resp. 2-Segal spaces).

sets



spaces

$$X: \Delta^{op} \rightarrow \text{Set}$$



$$X: \Delta^{op} \rightarrow \mathcal{S}_{\text{spaces}}$$

vector space with
 basis $B \in \text{Set}$



objects in the
 slice cat. \mathcal{S}/B

$$\mathcal{C}(X) = \bigoplus_{x \in X_1} k$$



$$\mathcal{C}(X) = \mathcal{S}/X_1$$

"free vector space
 on X_1 "

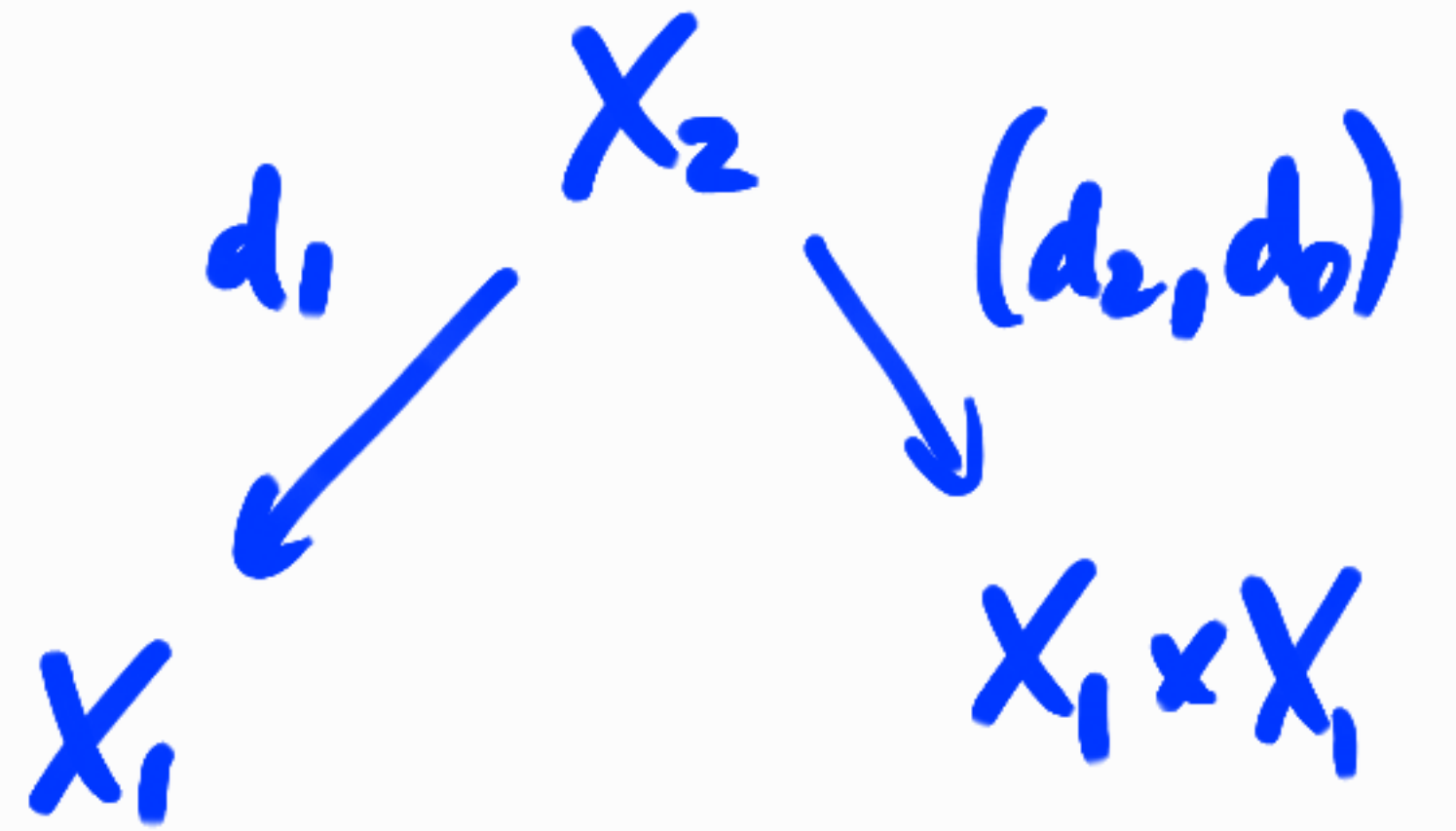
$$\Delta: \mathcal{C}(X) \rightarrow \mathcal{C}(X) \otimes \mathcal{C}(X)$$



$$\mathcal{S}/X_1 \rightarrow \mathcal{S}/X_1 \times X_1$$

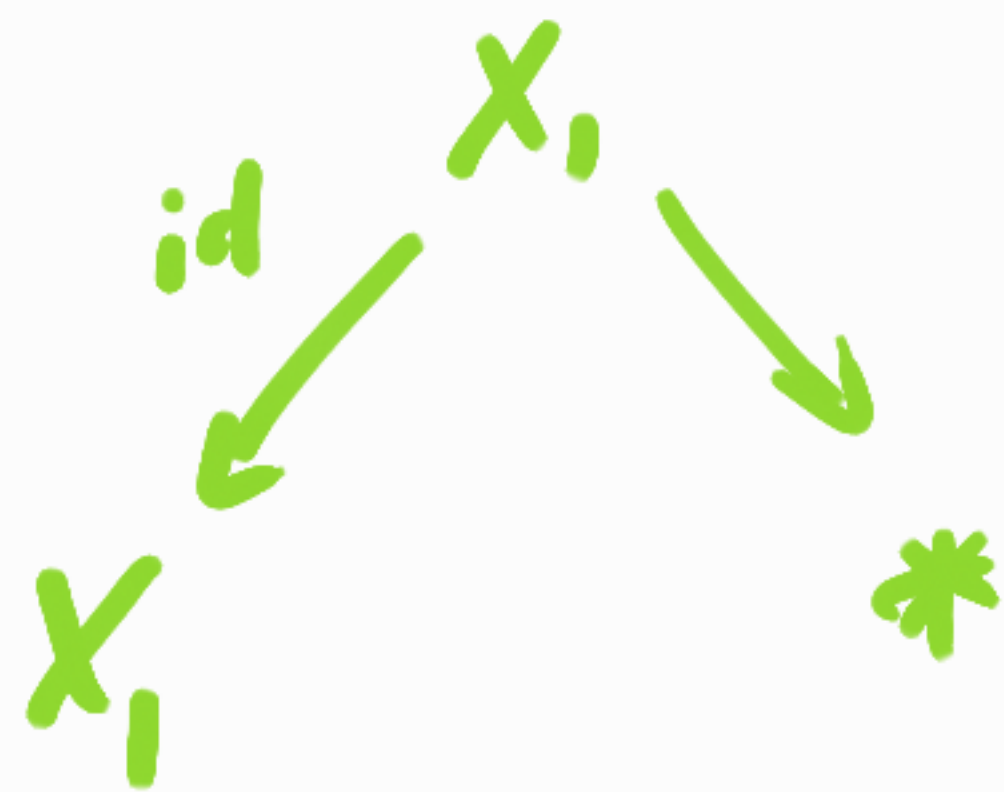
$$[x] \mapsto \sum_{d_1, d_2 = x} d_2 \otimes d_1$$

induced by



$$\mathfrak{f}: [x] \mapsto 1 \rightsquigarrow S/x_1 \xrightarrow{\mathfrak{f}} S$$

induced by



To recover the classical examples of zeta functions, take cardinality.

But there's so much more information available to us through decomposition spaces and homotopy theory!

Towards a decomposition space
description of Kapranov's
motivic zeta function

Recall: Kapranov's motivic zeta
function for a k -variety X
is the generating series

$$Z_{\text{mot}}(X, t) = \sum_{n=0}^{\infty} [\text{Sym}^n X] t^n \in K_0(\text{Var}_k)[[t]]$$

Key fact: when X is a variety
over $k = \mathbb{F}_q$,

$$Z(X, t) = \# Z_{\text{mot}}(X, t).$$

The Weil Conjectures imply $Z(-, t)$
extends to a motivic measure

$$K_0(\text{Var}_{\mathbb{F}_q}) \longrightarrow (1 + t \mathbb{Z}_\ell[[t]], \cdot)$$

$$[X] \longmapsto \prod_{i=0}^{2 \dim X} \det(1 - tF | H^i(X, \mathbb{Q}_\ell))^{(-1)^{i+1}}.$$

Campbell-Zakharovich construct a **lift**
of this measure to their K-theory

Spectra:

$$j: K(\text{Var}_{\mathbb{F}_q}) \rightarrow K(\text{Aut}(\mathbb{Z}_\ell))$$

"derived ℓ -adic zeta function"

(taking π_0 recovers $Z(-, t)$)

Fact: $Z_{\text{mot}}(-, t)$ is also a motivic measure:

$$Z_{\text{mot}}(-, t): K_0(\text{Var}_k) \rightarrow (1 + t K_0(\text{Var}_k)[[t]], \cdot)$$

Big Idea #1 (Campbell-Zakharovich)

$Z_{\text{mot}}(-, t)$ should lift to a map

$\mathcal{C} \rightarrow \mathcal{H}$ spectra

of K-theory spectra

$$I_{\text{mot}}: K(\text{Var}_k) \longrightarrow ???$$

Big Idea #2 (very much in progress
w/ Bergner, Feller)

$Z_{\text{mot}}(-, t)$ is the generating series
of the **zeta function** in the incidence
algebra of the decomposition space

$$\tilde{S}_0(\text{Var}_k)$$

Some evidence for this approach:

(not totally baseless speculation, I promise!)

- It works for $X = \text{Spec } k$:

the incidence algebra zeta function
is represented by the span

$$\text{iso}(\text{Var}_k) \xleftarrow{=} \text{iso}(\text{Var}_k) \longrightarrow *$$

which loosely corresponds to

$$[X] \longmapsto 1 \quad \text{for all } X$$

= [Spec k]

On the other hand,

$$Z_{\text{mot}}(\text{Spec } k, t) = (1-t)^{-1}$$

$$= \sum_{n=0}^{\infty} 1 t^n$$

- For any $X \in \text{Var}_k$, there is a product formula

$$Z_{\text{mot}}(X, t) = \prod_x (1-t)^{-1} = \prod_x Z_{\text{mot}}(\text{Spec } k(x), t)$$

so maybe we should really be treating

$Z_{\text{mot}}(X, t)$ as a "relative zeta function"

with respect to $X \rightarrow \text{Spec } k$.

Hodge poly.
of X/\mathbb{C}

for char.

Euler
of $X(\mathbb{C})$

$(k=\mathbb{C})$

$(k=\mathbb{C})$

$\tilde{S}_0(\text{Var}_k)$

$(k=\mathbb{F}_q)$
derived
 ℓ -adic

arithmetic
 $(k=\mathbb{Q})$

motivic

$K(\text{Var}_{\mathbb{Q}})$

$K(\text{Var}_{\mathbb{F}_q})$

$K(\text{Var}_k)$

\downarrow *arith*

\downarrow *g*

$\#$

\downarrow *mot*

$K(\text{Aut}(\mathbb{Z}))$

$K(\text{Aut}(\mathbb{Z}_\ell))$

$K(K(\text{Var}_k))$

Questions?

Bonus: there is a motivic

Möbius function

$$M_{\text{mot}}(X, t) = Z_{\text{mot}}(X, t) = \prod_{x \in X} (1 - t)$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n [\text{Conf}^n X] t^n$$

