Understanding Artin-Schreier Covers of Stacks

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Introduction

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Describe Artin-Schreier covers of curves using the language of stacks.

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Moduli problems are ‘the right framework’ to study modular forms – at least, for an algebraic geometer like me :)

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- Stacky curves

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$$\text{Modular forms of weight } k \longleftrightarrow \text{sections of the line bundle } \omega_X^{\otimes k}$$
$$\longleftrightarrow \Gamma\text{-invariant sections of } \omega_{\mathbb{C}}^{\otimes k}$$

where $f(z) \in \mathcal{M}_k \iff f(z)\, dz^k$ is Γ-invariant.

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There is a bijection

$$(L, s) \longleftrightarrow [\mathbb{A}^1/\mathbb{G}_m](X) = \mathrm{Hom}(X, [\mathbb{A}^1/\mathbb{G}_m])$$

where $L \to X$ is a line bundle with section s.

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Here, $[\mathbb{A}^1/\mathbb{G}_m]$ is an “infinitesimal thickening” of the classifying stack $B\mathbb{G}_m = [\bullet/\mathbb{G}_m]$.

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This has a natural answer in terms of Kummer theory.

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Definition (Cadman, '07)

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This allows one to study quotients of schemes in characteristic 0 as stacky objects.

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So to keep track of this additional arithmetic information, we need more structure than $[\mathbb{A}^1/\mathbb{G}_m] \approx$ point + fuzz.

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Definition

For $m \in \mathbb{Z}$, the **universal Artin-Schreier stack with jump m** is the cover of stacks

$$\begin{aligned}[\mathbb{P}(m,1)/\mathbb{G}_a] &\longrightarrow [\mathbb{P}(m,1)/\mathbb{G}_a] \\ [x,y] &\longmapsto [x^p - xy^{m(p-1)}, y^p]\end{aligned}$$

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This allows us to construct "Artin-Schreier roots" of line bundles as follows.

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For a stack X, a line bundle $L \to X$ and sections $s : X \to L, f : X \to L^{\otimes m}$, the **Artin-Schreier root stack** with jump m is the fibre product

$$\begin{array}{ccc} \wp_m^{-1}((L,s,f)/X) & \longrightarrow & [\mathbb{P}(m,1)/\mathbb{G}_a] \\ \downarrow & & \downarrow \Psi_m \\ X & \longrightarrow & [\mathbb{P}(m,1)/\mathbb{G}_a] \end{array}$$

where $X \to [\mathbb{P}(m,1)/\mathbb{G}_a]$ is the morphism determined by (L,s,f) and Ψ_m is the universal Artin-Schreier stack with jump m.

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An affine Artin-Schreier curve is of the form

$$Y = \mathrm{Spec}(k(x)[y]/(y^p - y - f(x)))$$

where $f(x)$ is a rational function of degree $m \in \mathbb{Z}, (m, p) = 1$.

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For $X = \mathbb{A}^1$, $L = \mathcal{O}_X, s = x$ and $f = f(x)$, we have

$$\wp_m^{-1}((\mathcal{O}_X, x, f)/X) \cong [Y/(\mathbb{Z}/p\mathbb{Z})]$$

where $\mathbb{Z}/p\mathbb{Z}$ acts additively.

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where $\mathbb{Z}/p\mathbb{Z}$ acts additively. Similar for any $X = \mathrm{Spec}\, A$.

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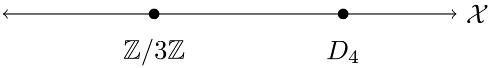

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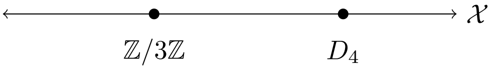

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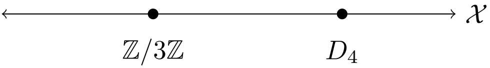

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Define the **canonical ring** of $\mathcal{X}$ to be

$$R(\mathcal{X}) = \bigoplus_{k=0}^{\infty} H^0(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k})$$

where $\omega_{\mathcal{X}}$ is the canonical sheaf on $\mathcal{X}$, defined in a similar way as that of a scheme.

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When $\mathcal{X} = \mathfrak{h}/\Gamma_0(N)$ for a congruence subgroup $\Gamma_0(N)$, then $H^0(\mathcal{X}, \omega_{\mathcal{X}}^{\otimes k}) \cong \mathcal{S}_{2k}(N)$, the space of weight $2k$ cusp forms of level N.

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Voight and Zureick-Brown: combinatorial description of $R(\mathcal{X})$ when $\mathcal{X}$ is tame, giving dimension formulas for spaces of modular forms in many cases.

Their proof uses: every tame stacky curve over an algebraically closed field is a root stack.

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Can $\wp_m^{-1}((L, s, f)/X)$ be generalized further to handle cyclic Galois groups of higher order? Nonabelian groups? (Artin-Schreier-Witt theory)

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For example, there exist stacky quotients of $X_0(p)$ with nonabelian stabilizers.

Thank you!

References

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- Voight, Zureick-Brown. The canonical ring of a stacky curve. (preprint)