Fundamental Groups in Algebra and Geometry

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Introduction

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- objects: covers $Y \xrightarrow{p} X$
- morphisms: $\mathrm{Hom}_X(Y, Z)$ is maps commuting with covers:

$$\begin{array}{ccc} Y & \longrightarrow & Z \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

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Unfortunately, $\widetilde{X}$ does not exist in algebraic categories, so we adopt a different perspective of $\pi_1^{top}(X)$.

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For a point $x \in X$, the **fibre functor** over x is

$$\begin{aligned} \mathrm{Fib}_x : \mathtt{Cov}_X &\longrightarrow \mathtt{Sets} \\ (Y \xrightarrow{p} X) &\longrightarrow p^{-1}(x). \end{aligned}$$

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$\mathrm{Aut}(\mathrm{Fib}_x)$ will be a good candidate for defining fundamental groups in other settings.

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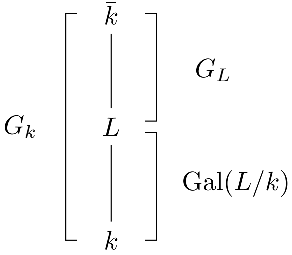

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Definition

A **morphism of schemes** is a pair $(f, f^\#) : (Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ consisting of:

(a) A map between spaces $f : Y \to X$

(b) A morphism of sheaves $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ that preserves local ring structure.

Étale covers

Definition

A morphism of schemes $p : Y \to X$ is **étale** at $y \in Y$ if the induced morphism on local rings $p^{\#} : \mathcal{O}_{X,p(y)} \to \mathcal{O}_{Y,y}$ satisfies:

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Let Fét_X be the category consisting of finite étale covers $Y \xrightarrow{p} X$, together with morphisms of covers:

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The fibre functor

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(Note that $\operatorname{Spec} k$ is, as a space, just a point. So this jives with the topological case.)

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The **étale fundamental group** of a scheme X at a geometric point $x : \operatorname{Spec} k \to X$ is defined as the automorphism group of the fibre functor over x:

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Theorem (Grothendieck)

For a connected scheme X and a geometric point $x : \operatorname{Spec} k \to X$,

(1) $\pi_1^{\text{ét}}(X, x)$ *is a profinite group which acts continuously on each fibre* $\operatorname{Fib}_x(Y)$ *for* $Y \to X$ *any cover.*

(2) Fib_x : *Fét*$_X \to \{$*continuous, finite, left* $\pi_1^{\text{ét}}(X, x)$*-sets*$\}$ *is an equivalence of categories.*

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Then

$$\left\{\begin{matrix}\text{finite étale covers} \\ Y \to \operatorname{Spec} k\end{matrix}\right\} \longleftrightarrow \{\text{finite étale } k\text{-algebras}\}$$

and $\pi_1^{\text{ét}}(\operatorname{Spec} k) \cong G_k = \operatorname{Gal}(k^{sep}/k)$.

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(You should think of $\operatorname{Spec} L$ for finite, separable extensions L/k as the connected covers of $\operatorname{Spec} k$.)

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There is an equivalence of categories

$$\left\{ \begin{array}{c} \textit{finite morphisms of curves} \\ Y \to X \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \textit{finitely generated field extensions} \\ L/k(X), \operatorname{tr}\deg_k L = 1 \end{array} \right\}$$

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$$\begin{array}{ccc} Y & \qquad & k(Y) \\ \downarrow & & \Big| \quad \operatorname{Gal}(k(Y)/k(X)) \\ X & & k(X) \end{array}$$

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Theorem

Let $k(X)_{ur}$ be the union of all field extensions $L/k(X)$ for L corresponding to a finite étale cover $Y \to X$. Then

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Proposition

Let $X/\mathbb{C}$ be a curve. Then the following are (anti-)equivalent:

(a) *finite morphisms of curves $Y \to X$*

(b) *finite extensions $L/k(X)$ where $\operatorname{tr}\deg_k L = 1$*

(c) *proper holomorphic maps of Riemann surfaces $Y(\mathbb{C}) \to X(\mathbb{C})$*

Curves: schemes of dimension 1

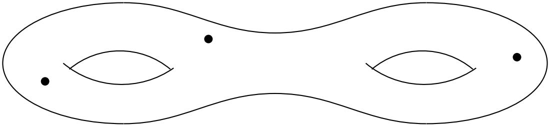
Theorem (Riemann Existence Theorem)

Let $X/\mathbb{C}$ be a curve and let

$$\pi_1^{top}(X(\mathbb{C})) = \left\langle x_1, y_1, \ldots, x_g, y_g, \gamma_1, \ldots, \gamma_n \,\middle|\, \prod_{i=1}^{g}[x_i, y_i] \prod_{j=1}^{n} \gamma_j = 1 \right\rangle$$

be the topological fundamental group of the underlying Riemann surface (g is the genus and n is the number of punctures). Then the étale fundamental group of X is the profinite completion of this group:

$$\pi_1^{\text{ét}}(X) \cong \widehat{\pi_1^{top}(X(\mathbb{C}))}.$$



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Theorem (Riemann Existence Theorem)

$$\pi_1^{\text{ét}}(X) \cong \pi_1^{top}\widehat{(X(\mathbb{C}))}$$

Example ($\mathbb{P}^1$)

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Any proper holomorphic map $Y \to \mathbb{P}^1$ is trivial $\implies \pi_1^{top}(X) = \{1\}$ $\implies \pi_1^{\text{ét}}(X) = \{1\} \implies$ any étale cover $Y \to \mathbb{P}^1$ is an isomorphism.

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$X = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C} \setminus \{0\}$, the punctured complex plane

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Example ($\mathbb{P}^1 \setminus \{0, \infty\}$)

$X = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C} \setminus \{0\}$, the punctured complex plane

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For every $n \in \mathbb{Z}$, there is a finite, degree n cover $X_n \to X$ corresponding to the holomorphic map

$$\begin{aligned} \mathbb{C} &\longrightarrow \mathbb{C} \setminus \{0\} \\ z &\longmapsto z^n. \end{aligned}$$

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This means every finite simple group is a quotient of $\pi_1^{\text{ét}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$!!! It is often said that this group is one of the most important in all of mathematics.

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† work with Lloyd West at University of Virginia

Thank you!