Wild Ramification and Stacky Curves

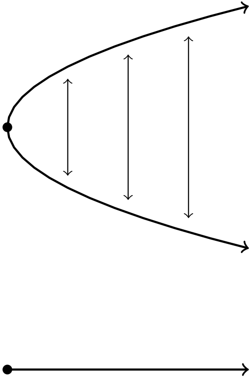
Andrew J. Kobin

ak5ah@virginia.edu

April 10, 2020

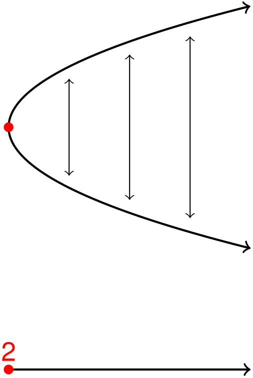
Introduction

Common problem: all sorts of information is lost when we consider quotient objects and/or singular objects.



Introduction

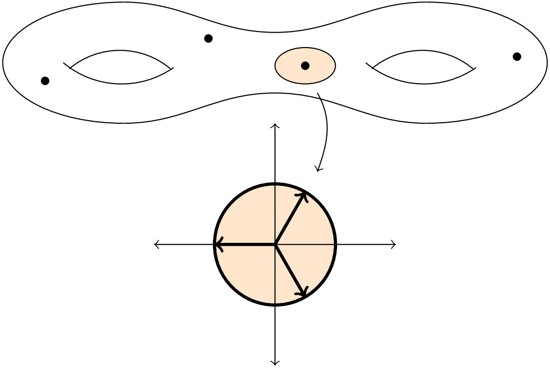
Solution: Keep track of lost information using *orbifolds* (topological and intuitive) or *stacks* (algebraic and fancy).



Complex Orbifolds

Definition

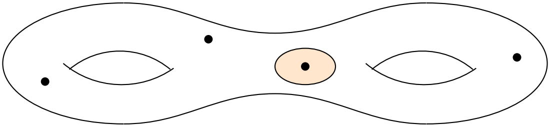
A **complex orbifold** is a topological space admitting an atlas $\{U_i\}$ where each $U_i \cong \mathbb{C}^n/G_i$ for a finite group G_i, satisfying compatibility conditions (think: manifold atlas but with extra info).



Algebraic Stacks

There's also a version of orbifold in algebraic geometry: an **algebraic stack**.

One important class of examples can be viewed as smooth varieties or schemes with a finite automorphism group attached at each point.



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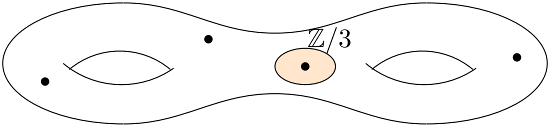

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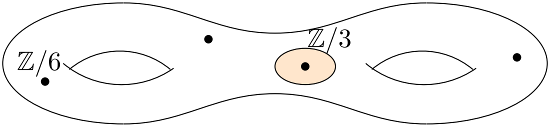

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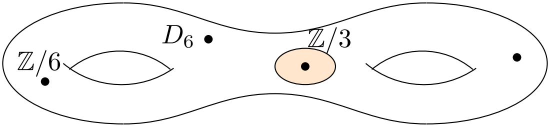

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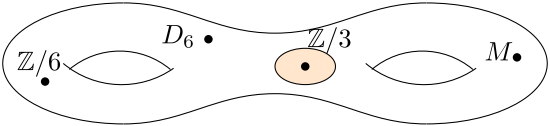

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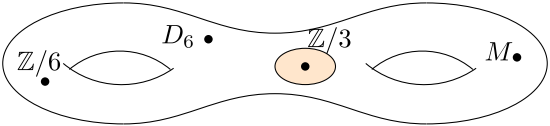
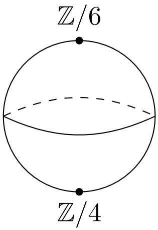


Focus on curves for the rest of the talk

An Example

Example

The (compactifed) moduli space of complex elliptic curves is a stacky $\mathbb{P}^1$ with a generic $\mathbb{Z}/2$ and a special $\mathbb{Z}/4$ and $\mathbb{Z}/6$.



Consequence: can deduce dimension formulas for modular forms from Riemann–Roch formula for stacks.

Goal: Classify stacky curves in char. p.

Main obstacle to overcome:

- In char. 0, local structure is determined by a cyclic group action.
- In char. p, this is not enough information – need more invariants than just the order of a cyclic group.

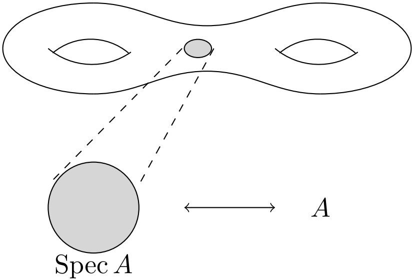
How we do it:

- Define local stacky structure intrinsically using line bundles and sections.
- Show this captures the local structure of a wild stacky curve.
- Fact: stacky curves come from local quotients.
- Show these quotients are linked to the intrinsic construction.

A Journey from Schemes to Stacks

For the uninitiated,

- An *affine scheme* is a topological space associated to a commutative ring.
- A *scheme* is a (locally ringed) space which is locally affine, or can be obtained by “gluing” affine schemes in a particular way.

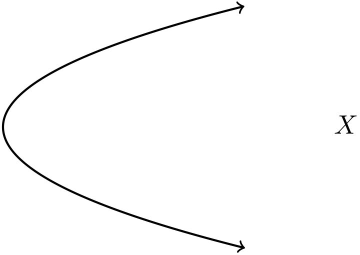


A Journey from Schemes to Stacks: The Functor of Points

To understand an object X (variety, scheme, etc.), it is enough to understand the set $X(T)$ of all maps $T \to X$ from all other objects T.

Example

Let X be a plane curve given by the equation $y^2 - x = 0$.



When $T = \operatorname{Spec} A$, $X(T) = \{$solutions to $y^2 - x = 0$ over $A\}$.

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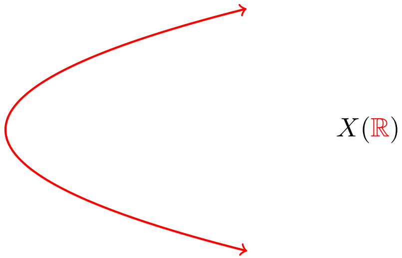

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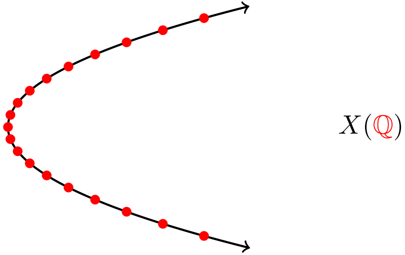

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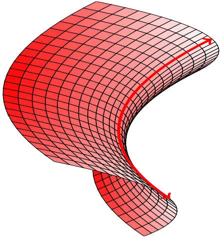


$X(\mathbb{C})$

When $T = \operatorname{Spec} A$, $X(T) = \{$solutions to $y^2 - x = 0$ over $A\}$.

A Journey from Schemes to Stacks: The Functor of Points

In other words, X determines a functor $X : \mathtt{AffSch}^{op} \to \mathtt{Set}$.

Consider instead a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ with values in the category of *groupoids*.

Motivation: the “points” in $X(T)$ may have nontrivial automorphisms, which can be recorded using groupoids instead of sets.

Definition

A **stack** is a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ satisfying *descent*: for any étale cover* $\{U_i \to T\}$, the objects/morphisms of $\mathcal{X}(T)$ correspond to compatible objects/morphisms of $\{\mathcal{X}(U_i)\}$.

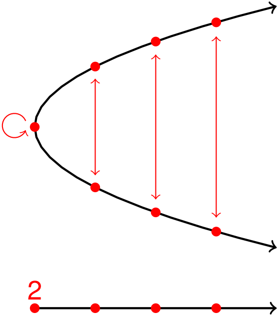
A Journey from Schemes to Stacks: The Functor of Points

Definition

A **stack** is a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ satisfying *descent*.

Example

For our plane curve $X : y^2 - x = 0$, groupoids remember automorphisms like $(x, y) \leftrightarrow (x, -y)$



Quotients

Example

Let Y be a curve and G be a group acting on Y. This determines a *quotient stack* $[Y/G] : \texttt{AffSch}^{op} \to \texttt{Gpd}$ defined by

$$[Y/G](T) = \left\{ \begin{array}{ccc} P & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \\ T & & \end{array} \right\}$$

where $p : P \to T$ is a principal G-bundle and $f : P \to Y$ is G-equivariant.

Stacky Curves

Definition

A **stacky curve** is a (smooth, separated, connected) stack $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ satisfying:

(1) $\mathcal{X}$ has an underlying *coarse moduli scheme* X with a map $\pi : \mathcal{X} \to X$ (collapse the groupoid to a set).

(2) π is an isomorphism away from a finite set of points.

(3) X is 1-dimensional (aka a curve).

(4) There is an étale surjection $U \to \mathcal{X}$ where U is a scheme.

(5) The diagonal $\Delta_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable.

Key facts:

- (4) $\implies$ each point of $\mathcal{X}$ has finitely many automorphisms.
- Locally about each point $x \in \mathcal{X}$, $\mathcal{X}$ looks like $[Y/G_x]$ where $G_x = \mathrm{Aut}(x)$.

Next: more on this local structure.

Root Stacks

Key fact: in char. 0, all stabilizers (automorphism groups) are *cyclic*.

So stacky curves can be locally modeled by a *root stack*: charts look like

$$U \cong [\operatorname{Spec} A/\mu_n]$$

where $A = K[y]/(y^n - \alpha)$ and μ_n is the group of nth roots of unity.

(Think: degree n branched cover mod μ_n-action, but remember the action using groupoids.)

Root Stacks

More rigorously:

Definition (Cadman, Abramovich–Olsson–Vistoli)

Let X be a scheme and $L \to X$ a line bundle with section $s : X \to L$. The **nth root stack** of X along (L, s) is the fibre product

$$\begin{array}{ccccc} \sqrt[n]{(L,s)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] & & x \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{(L,s)} & [\mathbb{A}^1/\mathbb{G}_m] & & x^n \end{array}$$

Here, $[\mathbb{A}^1/\mathbb{G}_m]$ is the classifying stack for pairs (L, s).

Interpretation: $\sqrt[n]{(L,s)/X}$ admits a canonical tensor nth root of (L, s), i.e. (M, t) such that $M^{\otimes n} = L$ and $t^n = s$ (after pullback).

Quick Break for Terminology

When our stacks are defined over a field k, we refer to them as:

- **tame** stacks if the stabilizer group of any point has order coprime to $\operatorname{char} k$ (always the case when $\operatorname{char} k = 0$);
- **wild** stacks if any stabilizer group has order divisible by $\operatorname{char} k$ (only happens when $\operatorname{char} k = p$ is prime).

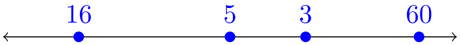
Root Stacks

Theorem (Geraschenko–Satriano '15)

Every smooth separated **tame** *Deligne–Mumford stack of finite type with trivial generic stabilizer is* * *a root stack over its coarse space.*

Corollary

Tame *stacky curves are completely described by their coarse space and a finite list of numbers corresponding to the orders of cyclic stabilizers at a finite number of stacky points.*



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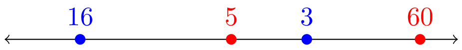

What happens with **wild** stacky curves in char. p?

Wild Stacky Curves

In trying to classify **wild** stacky curves in char. p, we face the following problems:

(1) Stabilizer groups need not be cyclic (or even abelian)

(2) Cyclic $\mathbb{Z}/p^n\mathbb{Z}$-covers of curves occur in families

(3) Root stacks don't work

- Finding $M^{\otimes p}$ is a problem
- $[\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m], x \mapsto x^p$ is a problem

Key case: cyclic $\mathbb{Z}/p\mathbb{Z}$ stabilizers

Artin–Schreier Theory

Idea: replace **tame** cyclic covers $y^n = f(x)$ with **wild** cyclic covers $y^p - y = f(x)$.

More specifically: Artin–Schreier theory describes cyclic degree p covers of curves:

$$\begin{array}{ccc} Y & : & y^p - y = f \\ \mathbb{Z}/p \downarrow & & \\ X & & \end{array}$$

- Every cyclic $\mathbb{Z}/p\mathbb{Z}$-cover of curves is birationally equivalent to one with equation $y^p - y = f$.
- At each pole of f, there is a **ramification jump** which is an invariant of the cover.
- Different jumps can yield non-isomorphic covers – this only happens in the **wild** case.
- Consequence: any classification of stacky curves must take the ramification jump into account.

Artin–Schreier Root Stacks

This suggests introducing wild stacky structure using the local model

$$U = [\operatorname{Spec} A/(\mathbb{Z}/p)]$$

where $A = K[y]/(y^p - y - f(x))$ and $\mathbb{Z}/p$ acts additively.

Artin–Schreier Root Stacks

How do we do it?

$$\begin{array}{ccc} \sqrt[n]{(L,s)/X} & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow & & \downarrow \\ X & \xrightarrow{(L,s)} & [\mathbb{A}^1/\mathbb{G}_m] \end{array} \qquad\qquad \begin{array}{c} x \\ \downarrow \\ x^n \end{array}$$

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$$\begin{array}{ccc} \sqrt[n]{(L,s)/X} & \longrightarrow & [\mathbb{P}^1/\mathbb{G}_a] \\ \downarrow & & \downarrow \\ X & \xrightarrow{(L,s)} & [\mathbb{P}^1/\mathbb{G}_a] \end{array} \qquad \begin{array}{c} [u,v] \\ \downarrow \\ [u^p, v^p - vu^{p-1}] \end{array}$$

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Artin–Schreier Root Stacks

How do we do it?

$$\begin{array}{ccc} \wp_1^{-1}((L,s,f)/X) & \longrightarrow & [\mathbb{P}^1/\mathbb{G}_a] \\ \downarrow & & \downarrow \\ X & \xrightarrow{(L,s,f)} & [\mathbb{P}^1/\mathbb{G}_a] \end{array} \qquad \begin{array}{c} [u,v] \\ \big\downarrow \\ [u^p, v^p - vu^{p-1}] \end{array}$$

Artin–Schreier Root Stacks

Definition (K.)

Fix $m \geq 1$. Let X be a scheme, $L \to X$ a line bundle and $s : X \to L$ and $f : X \to L^{\otimes m}$ two sections not vanishing simultaneously. The **Artin–Schreier root stack** of X with jump m along (L, s, f) is the normalized pullback

$$\begin{array}{ccc} \wp_m^{-1}((L,s,f)/X) & \longrightarrow & [\mathbb{P}(1,m)/\mathbb{G}_a] \\ \downarrow \; \nu & & \downarrow \\ X & \xrightarrow{(L,s,f)} & [\mathbb{P}(1,m)/\mathbb{G}_a] \end{array} \qquad \begin{array}{c} [u,v] \\ \downarrow \\ [u^p, v^p - vu^{m(p-1)}] \end{array}$$

where

- $\mathbb{P}(1, m)$ is the weighted projective line with weights $(1, m)$
- $\mathbb{G}_a = (k, +)$, acting additively
- $[\mathbb{P}(1, m)/\mathbb{G}_a]$ is the classifying stack for triples (L, s, f) up to the principal part of f.

Artin–Schreier Root Stacks

$$\begin{array}{ccc} \wp_m^{-1}((L,s,f)/X) & \longrightarrow & [\mathbb{P}(1,m)/\mathbb{G}_a] \\ \downarrow \quad \nu & & \downarrow \\ X & \xrightarrow{(L,s,f)} & [\mathbb{P}(1,m)/\mathbb{G}_a] \end{array} \qquad \begin{array}{c} [u,v] \\ \downarrow \\ [u^p, v^p - vu^{m(p-1)}] \end{array}$$

Interpretation: $\wp_m^{-1}((L,s,f)/X)$ admits a canonical pth root of L, i.e. a line bundle M such that $M^{\otimes p} = L$, and an AS root of s.

Artin–Schreier Root Stacks

Key example:

Example (K.)

Consider the AS cover

$$\begin{array}{c} Y \;:\; y^p - y = x^{-m} \\ \Big\downarrow \mathbb{Z}/p \\ \mathbb{P}^1 = \operatorname{Proj} k[x_0, x_1] \end{array}$$

where k is an algebraically closed field of characteristic p. Then

$$\wp_m^{-1}((\mathcal{O}(1), x_0, x_1^m)/\mathbb{P}^1) \cong [Y/(\mathbb{Z}/p)].$$

In general, every AS root stack is étale-locally isomorphic to such an "elementary AS root stack".

Artin–Schreier Root Stacks

Example (K.)

Let's see it for $m = 1$, so $Y : y^p - y = x^{-1}$.

- For a (local enough) test scheme T, $\wp_1^{-1}((\mathcal{O}(1), x_0, x_1)/\mathbb{P}^1)(T)$ consists of tuples (φ, L, s, f, ψ) where:
  - $\varphi : T \to \mathbb{P}^1$;
  - $L \to T$ is a line bundle with sections s and f;
  - $\psi : L^{\otimes p} \xrightarrow{\sim} \varphi^*\mathcal{O}(1)$ identifying $s^p = \varphi^* x_0$ and $f^p - f s^{p-1} = \varphi^* x_1$.
- $[Y/(\mathbb{Z}/p\mathbb{Z})](T)$ consists of diagrams

$$\left(\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \\ T & & \end{array} \right)$$

where $P \to T$ is a principal $\mathbb{Z}/p\mathbb{Z}$-bundle and $Y \to T$ is $\mathbb{Z}/p\mathbb{Z}$-equivariant.

Artin–Schreier Root Stacks

Example (K.)

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- For a test scheme T, $\wp_1^{-1}((\mathcal{O}(1), x_0, x_1)/\mathbb{P}^1)(T)$ consists of tuples (φ, L, s, f, ψ)
- $[Y/(\mathbb{Z}/p\mathbb{Z})](T)$ consists of diagrams

$$\left(\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \\ T & & \end{array} \right)$$

- The sections s, f allow one to build a $\mathbb{G}_a$-bundle $P \to T$ from L. Check: transition maps are in $\mathbb{Z}/p\mathbb{Z} \subseteq \mathbb{G}_a$.
- The equation $f^p - fs^{p-1} = \varphi^* x_1$ determines an equivariant map $P \to Y$.

Artin–Schreier Root Stacks

Example (K.)

Let's see it for $m = 1$, so $Y : y^p - y = x^{-1}$.

- This defines a functor $\wp_1^{-1}((\mathcal{O}(1), x_0, x_1)/\mathbb{P}^1)(T) \to [Y/(\mathbb{Z}/p\mathbb{Z})](T)$,

$$(\varphi, L, s, f, \psi) \longmapsto \left(\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \\ T & & \end{array} \right)$$

  for all T. Check: each one is an isomorphism.
- Finally, these assemble into an isomorphism of stacks $\wp_1^{-1}((\mathcal{O}(1), x_0, x_1)/\mathbb{P}^1) \xrightarrow{\sim} [Y/(\mathbb{Z}/p\mathbb{Z})]$.

Artin–Schreier Root Stacks

Useful properties of AS root stacks:

Proposition (K.)

(Naturality) If $h : Y \to X$ is a morphism then

$$\wp_m^{-1}((h^*L, h^*s, h^*f)/Y) \cong \wp_m^{-1}((L, s, f)/X) \times_X^\nu Y$$

where $\times_X^\nu$ denotes the normalized pullback.

In particular, AS root stacks can be iterated. Also:

Proposition (K.)

If $\mathcal{X}$ is a Deligne–Mumford stack (e.g. a stacky curve) then any AS root stack $\wp_m^{-1}((L, s, f)/\mathcal{X})$ is also Deligne–Mumford.

Upshot: can introduce **wild** stacky structure "from the ground up".

Classification of (Some) Wild Stacky Curves

So let's classify us some wild stacky curves!
(Assume: everything defined over $k = \bar{k}$)

Theorem 1 (K.)

Every Galois cover of curves $\varphi : Y \to X$ with an inertia group $\mathbb{Z}/p$ factors étale-locally through an Artin–Schreier root stack:

$$\begin{array}{ccccc} Y & & \xrightarrow{\quad\quad\varphi\quad\quad} & & X \\ \text{ét}\uparrow & & & & \uparrow\text{ét} \\ V & \longrightarrow & \wp_m^{-1}((L,s,f)/U) & \longrightarrow & U \end{array}$$

Informal consequence: there are infinitely many non-isomorphic stacky curves over $\mathbb{P}^1$ with a single stacky point of order p.

This phenomenon only occurs in char. p.

Classification of (Some) Wild Stacky Curves

Main result:

Theorem 2 (K.)

Every stacky curve $\mathcal{X}$ with a stacky point of order p is étale-locally isomorphic to an Artin–Schreier root stack $\wp_m^{-1}((L, s, f)/U)$ over an open subscheme U of the coarse space of $\mathcal{X}$.

This can even be done globally if $\mathcal{X}$ has coarse space $\mathbb{P}^1$:

Theorem 3 (K.)

If $\mathcal{X}$ has coarse space $\mathbb{P}^1$ and all stacky points of $\mathcal{X}$ have order p, then $\mathcal{X}$ is isomorphic to a fibre product of AS root stacks of the form $\wp_m^{-1}((L, s, f)/\mathbb{P}^1)$ for $(m, p) = 1$ and (L, s, f).

Classification of (Some) Wild Stacky Curves

Theorem 1 (K.)

Every Galois cover of curves $\varphi : Y \to X$ with an inertia group $\mathbb{Z}/p$ factors étale-locally through an Artin–Schreier root stack.

Proof Sketch: We want to find étale "neighborhoods" $U \to X$ and $V \to Y$ such that $\varphi|_V$ factors as $V \to \wp_m^{-1}((L, s, f)/U) \to U$.

- Arrange for $V \to U$ to be a one-point cover of curves with local equation $y^p - y = x^m$ (using Artin Approximation + result of Harbater on p-covers).
- Apply generalization of Key Example to get isomorphism of stacks $\wp_m^{-1}((L, s, f)/U) \cong [V/(\mathbb{Z}/p\mathbb{Z})]$.
- Since $\varphi|_V$ automatically factors through $[V/(\mathbb{Z}/p\mathbb{Z})]$, this gives the result.

□

Generalizations

What about $\mathbb{Z}/p^2$-covers, stacky points of order p^2, and beyond?

For cyclic stabilizer groups $\mathbb{Z}/p^n$, Artin–Schreier theory is subsumed by **Artin–Schreier–Witt theory**:

- AS equations $y^p - y = f(x)$ are replaced by Witt vector equations $\underline{y}^p - \underline{y} = \underline{f}(\underline{x}) = (f_0(\underline{x}), \ldots, f_n(\underline{x}))$.
- Covers are characterized by *sequences of ramification jumps*.
- Local structure is $U = [\operatorname{Spec} A/(\mathbb{Z}/p^n)]$ where $A = K[\underline{y}]/(\underline{y}^p - \underline{y} - \underline{f})$.

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What about $\mathbb{Z}/p^2$-covers, stacky points of order p^2, and beyond?

For cyclic stabilizer groups $\mathbb{Z}/p^n$, Artin–Schreier theory is subsumed by **Artin–Schreier–Witt theory**:

- AS equations $y^p - y = f(x)$ are replaced by Witt vector equations $\underline{y}^p - \underline{y} = \underline{f}(\underline{x}) = (f_0(\underline{x}), \ldots, f_n(\underline{x}))$.
- Covers are characterized by *sequences of ramification jumps*.
- Local structure is $U = [\operatorname{Spec} A/(\mathbb{Z}/p^n)]$ where $A = K[\underline{y}]/(\underline{y}^p - \underline{y} - \underline{f})$.

Question

How can we introduce structures like $[Y/(\mathbb{Z}/p^2\mathbb{Z})]$ intrinsically? Can these structures be classified?

Generalizations

This local structure can be formally introduced using **Artin–Schreier–Witt root stacks**:

$$\begin{array}{ccc} \wp_m^{-1}((L,s,f)/X) & \longrightarrow & [\mathbb{P}(1,m)/\mathbb{G}_a] \\ \downarrow \quad \nu \lrcorner & & \downarrow \wp_m \\ X & \xrightarrow{(L,s,f)} & [\mathbb{P}(1,m)/\mathbb{G}_a] \end{array}$$

where

- $\mathbb{G}_a$ is the additive group scheme
- $\mathbb{P}(1,m)$ is the weighted projective line
- $[\mathbb{P}(1,m)/\mathbb{G}_a]$ classifies triples (L,s,f) up to principal part of f.

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$$\begin{array}{ccc} \wp_m^{-1}((L,s,f)/X) & \longrightarrow & [\overline{\mathbb{W}}_n(\bar{m})/\mathbb{W}_n] \\ \downarrow \quad \nu \lrcorner & & \downarrow \Psi_{\bar{m}} \\ X & \xrightarrow{(L,s,f)} & [\overline{\mathbb{W}}_n(\bar{m})/\mathbb{W}_n] \end{array}$$

where

- $\mathbb{W}_n$ is the ring of length n Witt vectors
- $\overline{\mathbb{W}}_n(\bar{m})$ is a stacky compactification of $\mathbb{W}_n$ with weights $\bar{m} = (m_1, \ldots, m_n)$
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$$\begin{array}{ccc} \Psi_{\bar{m}}^{-1}(\varphi/X) & \longrightarrow & [\overline{\mathbb{W}}_n(\bar{m})/\mathbb{W}_n] \\ \downarrow \;\; \nu & & \downarrow \Psi_{\bar{m}} \\ X & \xrightarrow{\varphi} & [\overline{\mathbb{W}}_n(\bar{m})/\mathbb{W}_n] \end{array}$$

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(In progress) Next step is to classify stacky curves with $\mathbb{Z}/p^n$-structure using this construction.

Thank you!