Zeta functions and decomposition spaces

Based on:

A Primer on Zeta Functions and Decomposition Spaces

Andrew Kobin

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

Comments:	23 pages
Subjects:	**Number Theory (math.NT)**; Algebraic Geometry (math.AG); Category Theory (math.CT)
MSC classes:	11M06, 11M38, 14G10, 18N50, 16T10, 06A11, 55P99
Cite as:	arXiv:2011.13903 **[math.NT]**
	(or arXiv:2011.13903v1 **[math.NT]** for this version)

+ joint work in progress with Bogdan Krstic

Classical zeta functions

Riemann	Dedekind	Hasse-Weil
	$K/\mathbb{Q}$ # field	$X/\mathbb{F}_q$ variety
name:	name:	name:
$\zeta_{\mathbb{Q}}(s) = \sum^{\infty} \frac{1}{n^s}$	$\zeta_K(s) = \sum \frac{1}{N(\mathfrak{a})^s}$	$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$

product formula:	product formula:	product formula:
$\zeta_{\mathbb{Q}}(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$	$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$	$Z(X,t) = \prod_{x \in \lvert X \rvert} \left(1 - t^{\deg(x)}\right)^{-1}$
		cycle formula:
		$Z(X,t) = \sum_{\alpha} 1\, t^{\deg(\alpha)}$
underlying poset:	underlying poset:	underlying poset:
$(\mathbb{N}, \mid)$	$(I_K, \mid)$	$(Z_0^{\mathrm{eff}}(X), \le)$
arithmetic functions:	arithmetic functions:	arithmetic functions:
$f: \mathbb{N} \to \mathbb{C}$	$f: I_K \to \mathbb{C}$	$f: Z_0^{\mathrm{eff}}(X) \longrightarrow \mathbb{C}$
Dirichlet series:	Dirichlet series:	Dirichlet series:

$\sum n_\alpha \alpha \le \sum m_\alpha \alpha$ iff $n_\alpha \le m_\alpha$

series:	series:	
$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$	$\sum_{a} \frac{f(a)}{N(a)^s}$	$\sum_{\alpha} f(\alpha) t^{\deg(\alpha)}$
convolution:	convolution:	convolution:
$\left(\sum \frac{f(n)}{n^s}\right)\left(\sum \frac{g(n)}{n^s}\right) = \sum \frac{h(n)}{n^s}$	$\left(\sum \frac{f(a)}{N(a)^s}\right)\left(\sum \frac{g(a)}{N(a)^s}\right) = \sum \frac{h(a)}{N(a)^s}$	$\left(\sum f(\alpha) t^{\deg(\alpha)}\right)\left(\sum g(\alpha) t^{\deg(\alpha)}\right) = \sum h(\alpha) t^{\deg(\alpha)}$
$h(n) = \sum_{d \mid n} f(d) \cdot g(n/d)$	$h(a) = \sum_{d \mid a} f(d) \cdot g(a d^{-1})$	$h(\alpha) = \sum_{\beta \leq \alpha} f(\beta) g(\alpha - \beta)$
zeta function:	zeta function:	zeta function:
$\zeta : n \longmapsto 1$	$\zeta : a \longmapsto 1$	$\zeta : \alpha \longmapsto 1$
Möbius function:	Möbius function:	Möbius function:
$\mu : n \longmapsto$	$\mu : a \longmapsto$	$\mu : \alpha = \sum n_x x \longmapsto$
$= \begin{cases} 0, & p^2 \mid n \\ (-1)^r, & n = p_1 \cdots p_r \text{ distinct} \end{cases}$	$= \begin{cases} 0, & p^2 \mid a \\ (-1)^r, & a = p_1 \cdots p_r \text{ distinct} \end{cases}$	$= \begin{cases} 0, & n_x > 1 \\ (-1)^r, & \alpha = x_1 + \dots + x_r \text{ distinct} \end{cases}$
Möbius inversion:	Möbius inversion:	Möbius inversion:
$\mu * \zeta = 1$	$\mu * \zeta = 1$	$\mu * \zeta = 1$

$\mu * \zeta = 1$ | $\mu * \zeta = 1$ |

$\updownarrow$ | $\updownarrow$ | $\updownarrow$

$\zeta_{\mathbb{Q}}(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s}$ | $\zeta_K(s)^{-1} = \sum_{a} \frac{\mu(a)}{N(a)^s}$ | $Z(X,t)^{-1} = \sum_{\alpha} \mu(\alpha) t^{\deg(\alpha)}$

Zeta functions for posets

An interval in a (locally finite) poset $(P, \leq)$ is a sub-poset

$$[x,y] = \{ z \in P \mid x \leq z \leq y \}.$$

DEF The incidence coalgebra of a (locally finite) poset $(P, \leq)$ is the free k-vector space $C(P)$ on intervals, with : $\Delta : C(P) \longrightarrow C(P) \otimes C(P)$

$$[x,y] \longmapsto \sum_{z \in [x,y]} [x,z] \otimes [z,y]$$

DEF The incidence algebra of $(P, \leq)$ is the dual vector space

$$I(P) = \mathrm{Hom}_k(C(P), k)$$

with multiplication given by convolution:

$$I(P) \otimes I(P) \longrightarrow I(P)$$

$$\varphi \otimes \psi \longmapsto (\varphi * \psi)([x,y]) := \sum_{z \in [x,y]} \varphi([x,z])\psi([z,y])$$

In $I(P)$, there are distinguished elements: $\zeta : C(P) \longrightarrow k$

$$[x,y] \longmapsto 1$$

$$\mu : C(P) \longmapsto k$$

(recursive formula)

which satisfy $\mu * \zeta = 1 = \zeta * \mu$.

$\boxed{\text{Ex}}$ The Riemann, Dedekind and Hasse-Weil zeta functions, and their Möbius inversion principles, are all special cases.

Subtlety: need to pass to the <u>reduced</u> incidence algebra, e.g. $I(\mathbb{N}) \rightsquigarrow \tilde{I}(\mathbb{N})$

$[m,n] \rightsquigarrow \cancel{[1, \frac{n}{m}]}$

$[x,y] \cong [x',y'] \in P$

Fact: One can deduce the product formulas for $\zeta_{\mathbb{Q}}(s)$, $\zeta_K(s)$ and $Z(X,t)$

from a decomposition of the corresponding posets $\mathbb{N}$, I_K, $Z_0^{\text{eff}}(X)$.

$\mathbb{N} = \widetilde{\prod}\{p^k\}$

$\zeta = (1-\frac{1}{p})^{-1}$

$\boxed{\text{Decomposition spaces}}$

(due to Gálvez-Carrillo — Kock — Tonks)

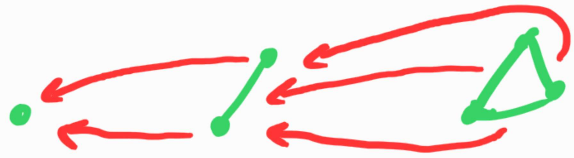
Idea: incidence algebras don't just come from posets, but <u>higher homotopy-theoretic structure</u>.

Recall: a **simplicial set** is a functor

$$X : \Delta^{op} \longrightarrow \text{Set}$$

Δ: cat. of combinatorial simplices

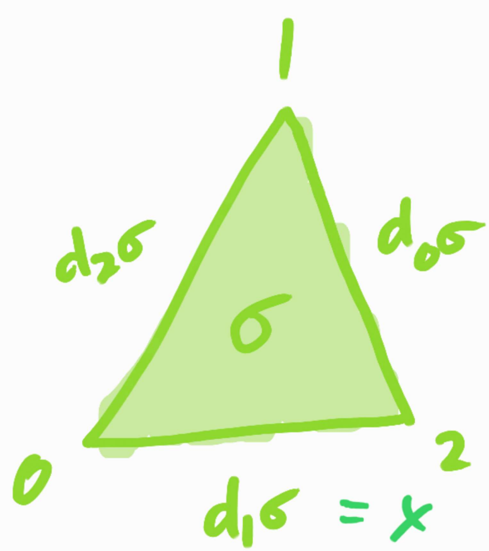
$$X_0 \rightleftarrows X_1 \quad X_2 \quad \cdots$$



DEF The **incidence coalgebra** of a (locally finite) simplicial set X is the k-vector space $C(X) = \bigoplus_{x \in X_1} k[x]$

with $\Delta : C(X) \longrightarrow C(X) \otimes C(X)$

$$[x] \longmapsto \sum_{\substack{\sigma \in X_2 \\ d_1\sigma = x}} d_2\sigma \otimes d_0\sigma.$$



Fact / "definition" $C(X)$ is a

coassociative, counital k-coalgebra exactly when X is a decomposition set (aka a 2-Segal set).

Gálvez-Carrillo–Kock–Tonks (and independently Dyckerhoff–Kapranov) generalize decomposition sets (resp. 2-Segal sets) to decomposition spaces (resp. 2-Segal spaces).

sets $\rightsquigarrow$ spaces

$X : \Delta^{op} \to Set$ $\rightsquigarrow$ $X : \Delta^{op} \to \mathcal{S}$ (spaces

vector space with basis $B \in Set$ $\rightsquigarrow$ objects in the slice cat. $\mathcal{S}_{/B}$

basis

$$C(X) = \bigoplus_{x \in X_1} k \rightsquigarrow C(X) = S_{/X_1}$$

"free vector space on X_1"

$$\Delta : C(X) \to C(X) \otimes C(X) \rightsquigarrow S_{/X_1} \to S_{/X_1 \times X_1}$$

$$[x] \mapsto \sum_{d_1\sigma = x} d_2\sigma \otimes d_1\sigma$$

induced by

$$\begin{tikzcd} & X_2 \arrow[dl, "d_1"'] \arrow[dr, "(d_2, d_0)"] & \\ X_1 & & X_1 \times X_1 \end{tikzcd}$$

$$\zeta : [x] \mapsto 1 \rightsquigarrow S_{/X_1} \xrightarrow{\zeta} S$$

induced by

$$\begin{tikzcd} & X_1 \arrow[dl, "id"'] \arrow[dr] & \\ X_1 & & * \end{tikzcd}$$

classical examples of zeta

To recover the classical examples of zeta functions, take cardinality.

But there's so much more information available to us through decomposition spaces and homotopy theory!

Towards a decomposition space description of Kapranov's motivic zeta function

---

$\boxed{\text{Def}}$ Kapranov's **motivic zeta function** for a k-variety X is the generating series

$$Z_{mot}(X,t)=\sum_{n=0}^{\infty}[\mathrm{Sym}^n X]t^n \in K_0(\mathrm{Var}_k)[[t]]$$

$$\Big\downarrow \#$$

$$\mathbb{Z}$$

$$\#[Y] := \#Y(\mathbb{F}_q)$$

Key fact: when X is a variety over $k=\mathbb{F}_q$,

$$Z(X,t)=\#Z_{mot}(X,t).$$

The Weil Conjectures imply $Z(-,t)$ extends to a motivic measure

$$K_0(\mathrm{Var}_{\mathbb{F}_q}) \longrightarrow (1+t\mathbb{Z}_\ell[[t]],\ \cdot\ )$$

$$[X] \longmapsto \prod_{i=0}^{2\dim X} \det\left(1-tF\,\middle|\,H^i(X,\mathbb{Q}_\ell)\right)^{(-1)^{i+1}}.$$