

Zeta functions and decomposition spaces

Based on:

A Primer on Zeta Functions and Decomposition Spaces

Andrew Kobin

Many examples of zeta functions in number theory and combinatorics are special cases of a construction in homotopy theory known as a decomposition space. This article aims to introduce number theorists to the relevant concepts in homotopy theory and lays some foundations for future applications of decomposition spaces in the theory of zeta functions.

Comments: 23 pages

Subjects: **Number Theory (math.NT)**; Algebraic Geometry (math.AG); Category Theory (math.CT)

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+ joint work in progress with Bogdan Krstic

Classical zeta functions

Riemann

name:

$$\zeta_{\mathbb{R}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

K/\mathbb{Q} #field

Dedekind

name:

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s}$$

X/\mathbb{F}_q variety

Hasse - Weil

name:

$$Z(X, t) = \exp \left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_q^n)}{n} t^n \right]$$

product formula:

$$\zeta_{\emptyset}(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}$$

product formula:

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$$

product formula:

$$Z(X, t) = \prod_{x \in |X|} \left(1 - t^{\deg(x)}\right)^{-1}$$

cycle formula:

$$Z(X, t) = \sum_{\alpha} 1 t^{\deg(\alpha)}$$

underlying poset:

$$(\mathbb{N}, |)$$

underlying poset:

$$(I_K, |)$$

underlying poset:

$$(Z_0^{\text{eff}}(X), \leq)$$

arithmetic functions:

$$f: \mathbb{N} \rightarrow \mathbb{C}$$

arithmetic functions:

$$f: I_K \rightarrow \mathbb{C}$$

arithmetic functions:

$$f: Z_0^{\text{eff}}(X) \rightarrow \mathbb{C}$$

Dirichlet series:

Dirichlet series:

Dirichlet series:

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

$$\sum_a \frac{f(a)}{N(a)^s}$$

$$\sum_{\alpha} f(\alpha) t^{\deg(\alpha)}$$

convolution:

convolution:

convolution:

$$\left(\sum \frac{f(n)}{n^s}\right) \left(\sum \frac{g(n)}{n^s}\right)$$

$$\left(\sum \frac{f(a)}{N(a)^s}\right) \left(\sum \frac{g(a)}{N(a)^s}\right)$$

$$\left(\sum f(\alpha) t^{\deg(\alpha)}\right) \left(\sum g(\alpha) t^{\deg(\alpha)}\right)$$

$$= \sum \frac{h(n)}{n^s}$$

$$= \sum \frac{h(a)}{N(a)^s}$$

$$= \sum h(\alpha) t^{\deg(\alpha)}$$

$$h(n) =$$

$$h(a) =$$

$$h(\alpha) =$$

zeta function:

zeta function:

zeta function:

$$f: n \mapsto 1$$

$$f: a \mapsto 1$$

$$f: \alpha \mapsto 1$$

Möbius function:

Möbius function:

Möbius function:

$$\mu: n \mapsto$$

$$\mu: a \mapsto$$

$$\mu: \alpha \mapsto$$

Möbius inversion:

Möbius inversion:

Möbius inversion:

$$\mu * f = 1$$

$$\mu * f = 1$$

$$\mu * f = 1$$

$$\begin{array}{ccc}
 \mu^* \uparrow = 1 & \mu^* \uparrow & \mu^* \uparrow \\
 \updownarrow & \updownarrow & \updownarrow \\
 \zeta_{\mathbb{Q}}(s)^{-1} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} & \zeta_K(s)^{-1} = \sum_a \frac{\mu(a)}{N(a)^s} & \zeta(X, t)^{-1} = \sum_{\alpha} \mu(\alpha) t^{\deg(\alpha)}
 \end{array}$$

Zeta functions for posets

An interval in a (locally finite) poset

(P, \leq) is a sub-poset

$$[x, y] = \{z \in P \mid x \leq z \leq y\}.$$

DEF The incidence coalgebra of a (locally finite) poset (P, \leq) is the free k -vector space $C(P)$ on intervals,

with: $\Delta: C(P) \rightarrow C(P) \otimes C(P)$

$$[x, y] \mapsto \sum_{z \in [x, y]} [x, z] \otimes [z, y]$$

DEF The incidence algebra of (P, \leq)

is the dual vector space

$$I(P) = \text{Hom}_k(C(P), k)$$

with multiplication given by convolution:

$$I(P) \otimes I(P) \rightarrow I(P)$$

$$\varphi \otimes \psi \mapsto (\varphi * \psi)([x, y]) :=$$

$$\sum_{z \in [x, y]} \varphi([x, z]) \psi([z, y])$$

In $I(P)$, there are distinguished

elements: $f: C(P) \rightarrow k$

$$[x, y] \mapsto 1$$

$$\mu : C(P) \longrightarrow k$$

(recursive formula)

which satisfy $\mu * \eta = 1 = \eta * \mu$.

Ex The Riemann, Dedekind and Hasse-Weil zeta functions, and their Möbius inversion principles, are all special cases.

Subtlety: need to pass to the reduced

incidence algebra, e.g. $I(N) \rightsquigarrow \tilde{I}(N)$

$$[m, n] \rightsquigarrow [1, \frac{n}{m}]$$

Fact: One can deduce the product formulas for $\zeta_G(s)$, $\zeta_K(s)$ and $Z(X, t)$

from a decomposition of the corresponding posets $N, I_K, Z_0^{\text{eff}}(X)$.

Decomposition spaces

(due to Gañez-Carrillo-Kock-Tonks)

Idea: incidence algebras don't just come from posets, but higher homotopy-theoretic structure.

Recall: a **simplicial set** is a functor

$$X: \Delta^{\text{op}} \longrightarrow \text{Set}$$

$$X_0 \begin{array}{c} \rightrightarrows \\ \leftleftarrows \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \end{array} X_1 \begin{array}{c} \rightrightarrows \\ \leftleftarrows \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \end{array} X_2 \begin{array}{c} \rightrightarrows \\ \leftleftarrows \\ \rightarrow \\ \leftarrow \\ \rightarrow \\ \leftarrow \end{array} \dots$$

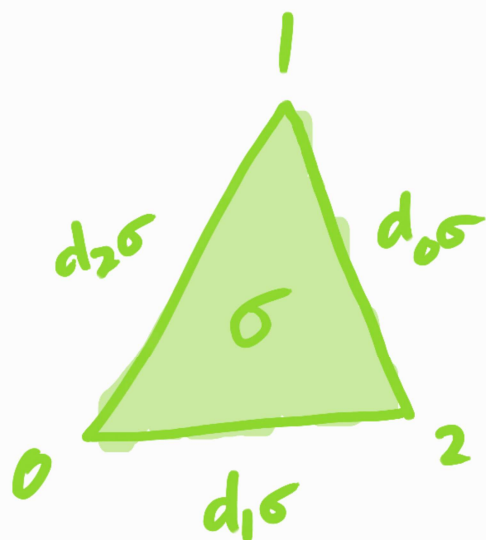
DEF The incidence coalgebra of a

(locally finite) simplicial set X is

the k -vector space $\mathcal{C}(X) = \bigoplus_{x \in X} k\{x\}$

with $\Delta : \mathcal{C}(X) \rightarrow \mathcal{C}(X) \otimes \mathcal{C}(X)$

$$[x] \mapsto \sum_{\substack{\sigma \in X_2 \\ d_1 \sigma = x}} d_2 \sigma \otimes d_0 \sigma.$$



Fact / "definition"

$\mathcal{C}(X)$ is a

coassociative, counital k -coalgebra
 exactly when X is a decomposition
 set (aka a 2-Segal set).

Gálvez-Carrillo - Kock - Tonks (and
 independently Dyckerhoff - Kapranov)
 generalize decomposition sets (resp.
 2-Segal sets) to decomposition
 spaces (resp. 2-Segal spaces).

sets



spaces

$$X: \Delta^{op} \rightarrow \text{Set}$$



$$X: \Delta^{op} \rightarrow \mathcal{S} \text{ (spaces)}$$

vector space with
 \mathbb{R} or \mathbb{C} as Set



objects in the
 slice cat. \mathcal{S}/\mathbb{R}

$$\mathcal{C}(X) = \bigoplus_{x \in X_1} k \rightsquigarrow$$

$$\mathcal{C}(X) = S/X_1$$

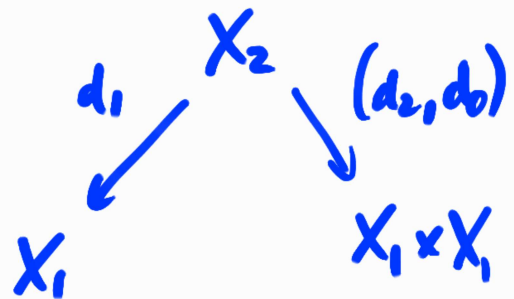
"free vector space
on X_1 "

$$\Delta: \mathcal{C}(X) \rightarrow \mathcal{C}(X) \otimes \mathcal{C}(X) \rightsquigarrow$$

$$[x] \mapsto \sum_{d, \sigma = x} d_2 \sigma \otimes d_1 \sigma$$

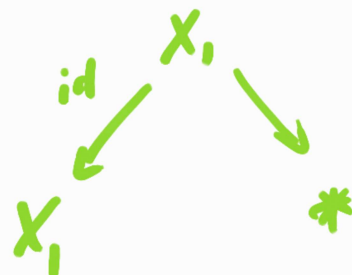
$$S/X_1 \rightarrow S/X_1 \times X_1$$

induced by



$$\zeta: [x] \mapsto 1 \rightsquigarrow S/X_1 \xrightarrow{\zeta} S$$

induced by



... classical examples of zeta

To recover the classical example of...

functions, take cardinality.

But there's so much more information available to us through decomposition spaces and homotopy theory!

Towards a decomposition space description of Kapranov's motivic zeta function

Def

Kapranov's motivic zeta function for a k -variety X

is the generating series

$$Z_{\text{mot}}(X, t) = \sum_{n=0}^{\infty} [\text{Sym}^n X] t^n \in K_0(\text{Var}_k[[t]])$$

Key fact: when X is a variety
over $k = \mathbb{F}_q$,

$$Z(X, t) = \# Z_{\text{mot}}(X, t).$$

The Weil Conjectures imply $Z(-, t)$
extends to a motivic measure

$$K_0(\text{Var}_{\mathbb{F}_q}) \longrightarrow (1 + t \mathbb{Z}_\ell[[t]], \cdot)$$

$$[X] \longmapsto \prod_{i=0}^{2 \dim X} \det(1 - tF | H^i(X, \mathbb{Q}_\ell))^{(-1)^{i+1}}.$$