Campbell – Wolfson – Zakharevich construct a lift of this measure to their K-theory spectra:

$$\zeta : K(\mathrm{Var}_{\mathbb{F}_q}) \longrightarrow K(\mathrm{Aut}(\mathbb{Z}_\ell))$$

"derived ℓ-adic zeta function"

(taking π_0 recovers $Z(-,t)$)

Fact: $Z_{mot}(-,t)$ is also a motivic measure:

$$Z_{mot}(-,t) : K_0(\mathrm{Var}_k) \longrightarrow (1 + t K_0(\mathrm{Var}_k)[[t]], \cdot\,).$$

(Campbell–Zakharevich)

$\boxed{\text{Big Idea \#1}}$ (Campbell [illegible])

$Z_{mot}(-,t)$ should lift to a map of K-theory spectra

$$\zeta_{mot} : K(Var_k) \longrightarrow ???$$

$\boxed{\text{Big Idea \#2}}$ (very much in progress w/ Krstic)

$Z_{mot}(-,t)$ is the generating series of the zeta function in the incidence algebra of the decomposition space

$$\tilde{S}_\bullet(Var_k)$$

Some evidence for this approach:
(not totally baseless speculation, I promise!)

- It works for $X = \operatorname{Spec} k$:

  the incidence algebra zeta function is represented by the span

$$\mathrm{iso}(\mathrm{Var}_k) \xleftarrow{=} \mathrm{iso}(\mathrm{Var}_k) \longrightarrow *$$

which loosely corresponds to

$$[X] \longmapsto \underset{\substack{\| \\ [\operatorname{Spec} k]}}{1} \quad \text{for all } X$$

On the other hand,

$$Z_{mot}(\operatorname{Spec} k, t) = (1-t)^{-1}$$
$$= \sum_{n=0}^{\infty} 1\, t^n$$

- For any $X \in Var_k$, there is a product formula

$$Z_{mot}(X,t) = \prod_x (1-t)^{-1} = \prod_x Z_{mot}(\operatorname{Spec} k(x), t)$$

so maybe we should really be treating $Z_{mot}(X,t)$ as a "relative zeta function" with respect to $X \to \operatorname{Spec} k$.

Hodge poly.

of $X/\mathbb{C}$

of $X(\mathbb{C})$ $\xleftarrow{(k=\mathbb{C})}$ $\widetilde{S}_0(\mathrm{Var}_k)$ $\xrightarrow{(k=\mathbb{C})}$ Euler char. of $X(\mathbb{C})$

$$\widetilde{S}_0(\mathrm{Var}_k) \xrightarrow[(k=\mathbb{F}_q)]{\text{derived } \ell\text{-adic}} K(\mathrm{Var}_{\mathbb{F}_q}) \xrightarrow{\zeta} K(\mathrm{Aut}(\mathbb{Z}_\ell))$$

$$\widetilde{S}_0(\mathrm{Var}_k) \xrightarrow{\text{motivic}} K(\mathrm{Var}_k) \xrightarrow{\zeta_{mot}} K(K(\mathrm{Var}_k))$$

$$\widetilde{S}_0(\mathrm{Var}_k) \xrightarrow[(k=\mathbb{Q})]{\text{arithmetic}} K(\mathrm{Var}_{\mathbb{Q}}) \xrightarrow{\zeta_{arith}} K(\mathrm{Aut}(\mathbb{Z}))$$

$$K(K(\mathrm{Var}_k)) \xrightarrow{\#} K(\mathrm{Aut}(\mathbb{Z}_\ell))$$

Questions?

Bonus: there is a motivic Möbius function

$$M_{mot}(X,t) = Z_{mot}(X,t)^{-1} = \prod_{x \in X} (1-t)$$

$$= 1 + \sum_{n=1}^{\infty} (-1)^n [\mathrm{Conf}^n X] t^n$$