What is a Stack? Part 1

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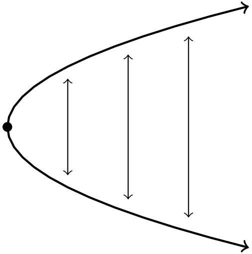
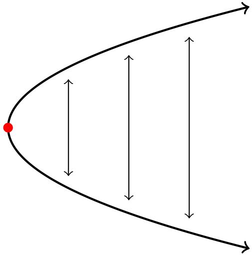
Introduction

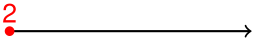
Common problem: all sorts of information is lost when we consider quotient objects and/or singular objects.



Introduction

Solution: Keep track of lost information using *orbifolds* (topological and intuitive) or *stacks* (algebraic and fancy).

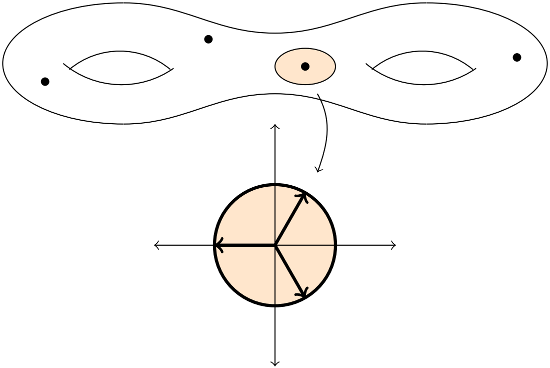




Complex Orbifolds

Definition

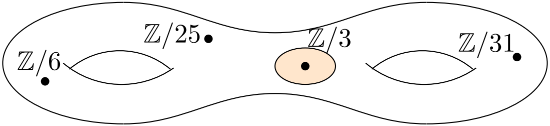
A **complex orbifold** is a topological space admitting an atlas $\{U_i\}$ where each $U_i \cong \mathbb{C}^n/G_i$ for a finite group G_i, satisfying compatibility conditions (think: manifold atlas but with extra info).



Algebraic Stacks

There's also a version of orbifold in algebraic geometry: an **algebraic stack**.

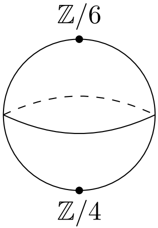
One important class of examples (Deligne–Mumford stacks) can be viewed as smooth varieties or schemes with a finite automorphism group attached at each point.



Running Example

Example

The (compactifed) moduli space of complex elliptic curves is a stacky $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ with a generic $\mathbb{Z}/2$ and a special $\mathbb{Z}/4$ and $\mathbb{Z}/6$.



We will delve further into this example in a bit.

For now: "algebraic objects can be parametrized algebraically", but we should keep track of automorphisms (e.g. to count properly!)

Moduli Problems

Loosely, a moduli problem is of the form "classify all objects of a certain type" admitting some natural geometric structure.

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Examples:

- circles $\longleftrightarrow$ radius $+$ center
- circles up to isometry $\longleftrightarrow$ radius
- lines through the origin $\longleftrightarrow$ unit vector / $\pm$
- subspaces of a vector space $\longleftrightarrow$ r-frame / invertible matrix
- hyperbolic structures on Σ_g $\longleftrightarrow$ $6g-6$ coordinates
- lines on a cubic surface $\longleftrightarrow$ there are exactly 27
- more examples?

Moduli Problems

More precisely, a moduli problem takes the form of a functor $P : \texttt{Schemes} \to \texttt{Sets}$.

We say P is **representable** if $P(X) = \mathrm{Hom}(X, M)$ for some space M – called a **moduli space** for P.

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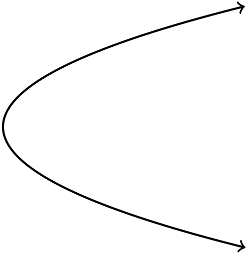
- hyperbolic structures on $\Sigma_g \longleftrightarrow \mathbb{R}^{6g-6}$
- lines on a cubic surface $\longleftrightarrow$ zero-dim. variety with 27 points
- more examples?

Moduli Problems

Many moduli problems are algebraic in nature ("algebraic objects classified algebraically").

Example

The solutions to a polynomial equation $f(x_1, \ldots, x_n) = 0$ correspond to the points of an *algebraic variety*:



$$y^2 - x = 0$$

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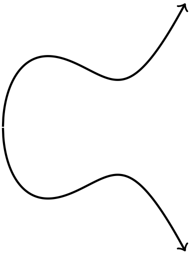

$$y^2 - x^3 + x - 1 = 0$$

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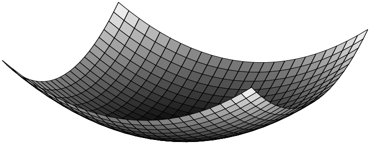

$$x^2 + y^2 - z - 1 = 0$$

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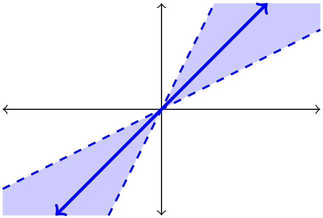
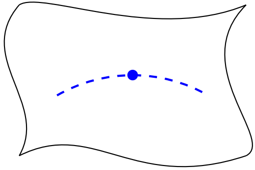
Example

The moduli problem of finding r-dimensional subspaces of an n-dimensional vector space V is *represented* by a variety $\mathrm{Gr}(r, n)$ called the Grassmannian variety

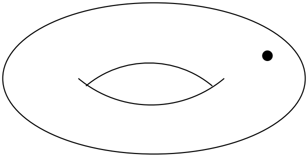
e.g. $\mathrm{Gr}(1, n) = \mathbb{P}^n$, projective n-space.





Running Example: Elliptic Curves

An **elliptic curve** is a 1-dimensional complex manifold E (a *Riemann surface*) of genus 1 together with a specified basepoint O.



Fact: Every complex elliptic curve can be described by a *Weierstrass equation* $y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{C}$.

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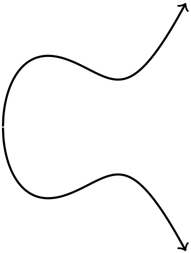

$y^2 = x^3 - x + 1$

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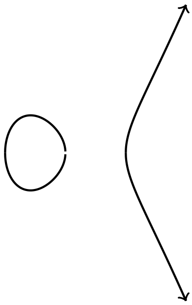

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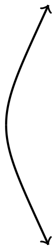

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To classify elliptic curves, we define an invariant

$$j(E) = 1728 \frac{4A^3}{4A^3 + 27B^2} \in \mathbb{C}.$$

Theorem

Two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$.

Corollary

The affine j-line $\mathbb{A}^1_j := \mathbb{C}$ is a moduli space for isomorphism classes of elliptic curves.

Running Example: Elliptic Curves

$$E : y^2 = x^3 + Ax + B, \quad j(E) = 1728\frac{4A^3}{4A^3 + 27B^2}$$

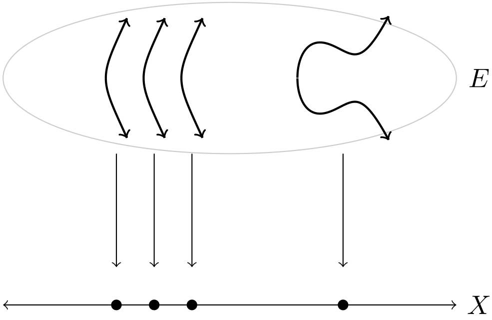
Corollary

The affine j-line $\mathbb{A}^1_j := \mathbb{C}$ is a moduli space for isomorphism classes of elliptic curves.

`https://www.desmos.com/calculator/ialhd71we3`

Running Example: Elliptic Curves

A **family of elliptic curves** over a complex manifold X is a holomorphic map $E \to X$ whose fibres are elliptic curves (and the distinguished points are picked out by a holomorphic section $O : X \to E$).



Running Example: Elliptic Curves

Let $\mathscr{M}_{1,1}(X)$ denote the set of isomorphism classes of (families of) elliptic curves over X. For any holomorphic map $Y \to X$, we get a map $\mathscr{M}_{1,1}(X) \to \mathscr{M}_{1,1}(Y)$ defined by pullback:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \big\downarrow & & \big\downarrow \\ Y & \longrightarrow & X \end{array}$$

That is, $\mathscr{M}_{1,1}$ defines a functor $\texttt{CxMfld}^{op} \to \texttt{Set}$.

Running Example: Elliptic Curves

Question: Is $\mathscr{M}_{1,1} : \texttt{CxMfld}^{op} \to \texttt{Set}$ *representable*? That is, $\mathscr{M}_{1,1}(-) \cong \mathrm{Hom}(-, M)$ for a complex manifold M?

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Answer: Unfortunately, no. If there was, there would be a "universal elliptic curve" $E_0 \in \mathscr{M}_{1,1}(M)$ corresponding to $id_M \in \mathrm{Hom}(M, M)$ such that every elliptic curve $E \to X$ would be a pullback

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E_0 \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & M \end{array}$$

for unique f and $\tilde{f}$.

However, no map $\tilde{f} : E \to E_0$ is unique: every elliptic curve $E : y^2 = x^3 + Ax + B$ has a nontrivial degree 2 automorphism $(x, y) \mapsto (x, -y)$. Further, if $j(E) = 0$ or 1728, $\mathrm{Aut}(E)$ is even larger.

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Partial Answer: The j-line $\mathbb{A}^1_j$ is a *coarse moduli space* for the moduli problem of elliptic curves. That is:

- there is a bijection $\mathscr{M}_{1,1}(*) \cong \mathrm{Hom}(*, \mathbb{A}^1_j) = \mathbb{C}$; and
- for any other complex manifold M and natural transformation $\mathscr{M}_{1,1}(-) \to \mathrm{Hom}(-, M)$, there is a unique map $\mathbb{A}^1_j \to M$ making the following diagram commute:

$$\begin{array}{ccc} \mathscr{M}_{1,1} & \longrightarrow & \mathrm{Hom}(-, \mathbb{A}^1_j) \\ & \searrow \quad \swarrow & \\ & \mathrm{Hom}(-, M) & \end{array}$$

Example from Topology

Here's a related example from topology.

Example

Let G be a group and consider the principal G-bundle functor

$$\texttt{Top} \longrightarrow \texttt{Set}, \quad X \longmapsto \mathrm{Bun}_G(X) = \{\text{principal } G\text{-bundles}\}/\text{iso}.$$

There is a space BG which classifies G-bundles up to isomorphism:

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Here, $(f, \tilde{f})$ are unique up to homotopy, i.e. there is a natural isomorphism $\mathrm{Bun}_G(-) \cong [-, BG]$. Want: unique on the nose.

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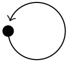
Example

To fix this, treat BG as a *groupoid*



$g \in G$

$\mathrm{Bun}_G(X)$ can also be a viewed as a groupoid (remember the isomorphisms $P \xrightarrow{\sim} P'$ between G-bundles over X).

Want a natural isomorphism

$$\mathrm{Bun}_G(-) \cong \mathrm{Hom}_{\texttt{Gpd}}(-, BG)$$

A Journey from Schemes to Stacks

The spaces I want to consider can be "described by algebra".

Motivation: Hilbert's Nullstellensatz says that the points of $\mathbb{C}^n$ are in bijection with certain ideals in a polynomial ring:

$$\mathbb{A}^n_{\mathbb{C}} := \mathbb{C}^n \longleftrightarrow \operatorname{MaxSpec} \mathbb{C}[z_1, \ldots, z_n]$$
$$P = (\alpha_1, \ldots, \alpha_n) \longmapsto \mathfrak{m}_P = (z_1 - \alpha_1, \ldots, z_n - \alpha_n).$$

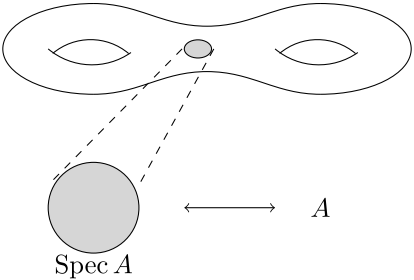
This is the jumping off point for algebraic geometry: for a general commutative ring A, there is a space $\operatorname{Spec} A$ defined by replacing "maximal ideals" with "prime ideals":

- Points of $\operatorname{Spec} A$ = prime ideals $\mathfrak{p} \subset A$
- Closed subsets of $\operatorname{Spec} A$ = "vanishing sets" $V(I) = \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\}$
- Sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} A$ with $\mathcal{O}(\operatorname{Spec} A) \cong A$ and stalks $\leftrightarrow$ localizations.

A Journey from Schemes to Stacks

Just as manifolds can be glued together from locally trivial patches, schemes are what we get by gluing together spaces built out of rings.

- An *affine scheme* is the space $\operatorname{Spec} A$ associated to a commutative ring A, together with its structure sheaf $\mathcal{O}$.
- A *scheme* is a locally ringed space which is locally affine, or can be obtained by "gluing" affine schemes (and their structure sheaves) in a compatible way.



Interlude: "Topology" is Flexible

Important: the gluing must play nicely with the chosen *topology* on $\operatorname{Spec} A$.

e.g. in ordinary topology, a cover $\{U_i \to X\}$ is a collection of open subsets $U_i \subseteq X$ with $\bigcup U_i = X$.

e.g. in the étale topology, a cover $\{U_i \to X\}$ is a collection of *étale morphisms* $f_i : U_i \to X$ with $\bigcup f_i(U_i) = X$.

($f : U \to X$ is *étale* if it induces an isomorphism on tangent spaces. That is, étale is the algebraic analogue of local homeomorphism in topology.)

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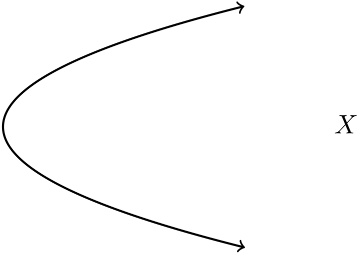
More on this next time...

A Journey from Schemes to Stacks: The Functor of Points

To understand an object X (variety, scheme, etc.), it is enough to understand the set $X(T)$ of all maps $T \to X$ from all other objects T.

Example

Let X be a plane curve given by the equation $y^2 - x = 0$.



When $T = \operatorname{Spec} A$, $X(T) = \{$solutions to $y^2 - x = 0$ over $A\}$.

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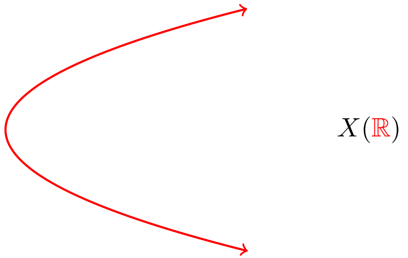

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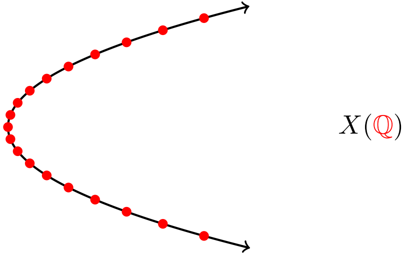

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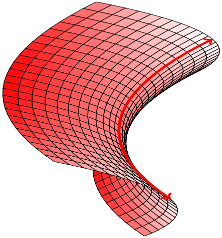

$X(\mathbb{C})$

When $T = \operatorname{Spec} A$, $X(T) = \{$solutions to $y^2 - x = 0$ over $A\}$.

A Journey from Schemes to Stacks: The Functor of Points

In other words, X determines a functor $X : \mathtt{AffSch}^{op} \to \mathtt{Set}$.

So every scheme X is like a moduli problem (super interesting in general: what do its points classify?)

Like we did with moduli problems, what happens if we instead have a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ with values in the category of *groupoids*.

Definition

A **stack** is a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ satisfying *descent*: for any étale cover* $\{U_i \to T\}$, the objects/morphisms of $\mathcal{X}(T)$ correspond bijectively to compatible objects/morphisms of $\{\mathcal{X}(U_i)\}$.

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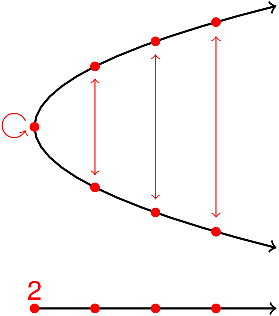
A **stack** is a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ satisfying *descent*.

Example

For our plane curve $X : y^2 - x = 0$, groupoids remember automorphisms like $(x, y) \leftrightarrow (x, -y)$



Quotients

Example

Let Y be a scheme and G be a group acting on Y. This determines a *quotient stack* $[Y/G] : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ defined by

$$[Y/G](T) = \left\{ \begin{array}{ccc} P & \xrightarrow{f} & Y \\ {\scriptstyle p}\big\downarrow & & \\ T & & \end{array} \right\}$$

where $p : P \to T$ is a principal G-bundle and $f : P \to Y$ is G-equivariant.

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Exercise: Why is this the right notion of quotient?

Quotients

Example

The *classifying stack* of a group G is the stack $BG : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ defined by

$$BG(T) = \text{the groupoid of principal } G\text{-bundles } P \to T.$$

Quotients

Example

The *classifying stack* of a group G is the stack $BG : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ defined by

$$BG(T) = \text{the groupoid of principal } G\text{-bundles } P \to T.$$

Alternatively, $BG = [\bullet/G]$ and the universal G-bundle from topology is replaced by:

$$\begin{array}{ccc} P & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ T & \longrightarrow & BG \end{array}$$

Next Time

Next time, we will:

- Get a better idea of étale covers and descent
- See more examples of stacks
- Talk more about elliptic curves
- Show that the elliptic curve moduli problem $\mathscr{M}_{1,1}$ is (represented by) a stack
- Cast a sideways glance at moduli spaces of other things (higher genus curves, abelian varieties, ??)

Thank you!

https://www.desmos.com/calculator/ialhd71we3