What is a Stack? Part 2

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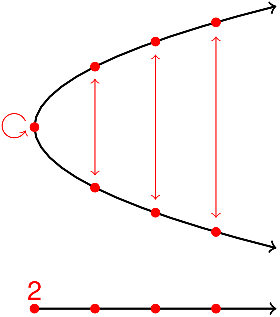
April 8, 2021

Recap

Our goal was to keep track of extra geometric information (e.g. nontrivial automorphisms, quotient data) using stacks.



Recap

More precisely, we considered **moduli problems** as functors $P : \texttt{Schemes} \to \texttt{Gpd}$ and asked if they were representable by a **moduli space** M:

$$P(-) \cong \mathrm{Hom}(-, M)$$

or something more general...

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...such as:

Definition

A **stack** is a functor $\mathcal{X} : \texttt{AffSch}^{op} \to \texttt{Gpd}$ satisfying *descent*: for any étale cover* $\{U_i \to T\}$, the objects/morphisms of $\mathcal{X}(T)$ correspond bijectively to compatible objects/morphisms of $\{\mathcal{X}(U_i)\}$.

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More about étale covers and descent in a moment, but recall the following key examples:

Example

Quotient stacks $[Y/G]$, for a group G acting on a scheme Y.

Example

The classifying stack $BG = [\bullet/G]$ of a group.

Recap

Example

Let Y be a scheme and G be a group acting on Y. This determines a *quotient stack* $[Y/G] : \texttt{AffSch}^{op} \to \texttt{Gpd}$ defined by

$$[Y/G](T) = \left\{ \begin{array}{ccc} P & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \\ T & & \end{array} \right\}$$

where $p : P \to T$ is a principal G-bundle and $f : P \to Y$ is G-equivariant.

Recap

Example

The *classifying stack* of a group G is the stack $BG : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ defined by

$$BG(T) = \text{the groupoid of principal } G\text{-bundles } P \to T.$$

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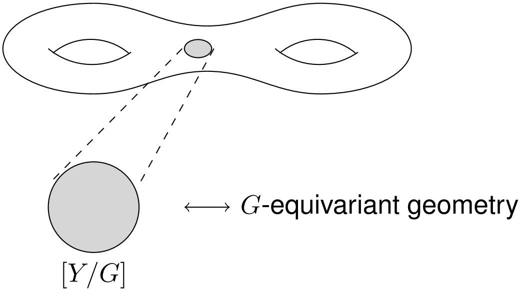
$$BG(T) = \text{the groupoid of principal } G\text{-bundles } P \to T.$$

Alternatively, $BG = [\bullet/G]$ and the universal G-bundle from topology is replaced by:

$$\begin{array}{ccc} P & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ T & \longrightarrow & BG \end{array}$$

Recap (not really)

Super Important Fact: Many classes of stacks are locally quotient stacks.



Étale Morphisms

One of the difficult (= fun) parts of algebraic geometry is to translate useful concepts from topology and differential geometry into algebra. For example:

- The **tangent space** to a scheme X at a point $x \in X$ is the dual vector space $T_x X = (\mathfrak{m}_x / \mathfrak{m}_x^2)^*$.
- A map $f : Y \to X$ of schemes is **smooth** if it is finite, flat and each fibre $Y_x = Y \times_X \{x\}$ is **regular**, i.e. $\dim Y_x = \dim T_y Y_x$ for all $y \in Y_x$.
- A smooth map $f : Y \to X$ is **étale** if each $\dim Y_x = 0$.

Loosely, étale morphisms are the algebro-geometric analogues of local homeomorphisms in differential geometry.

Étale Morphisms

Example

When $X = \mathbb{A}^n = \operatorname{Spec} k[t_1, \ldots, t_n]$ is affine n-space, $x = (\alpha_1, \ldots, \alpha_n)$,

- $T_x\mathbb{A}^n = \operatorname{Span}_k \left\{ \frac{\partial}{\partial(t_i - \alpha_i)} \right\}_{i=1}^n$ which has dim. n, so $\mathbb{A}^n$ is regular.
- The structure map $\mathbb{A}^n \to *$ is smooth.

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- The structure map $\mathbb{A}^n \to *$ is smooth.

More generally:

Proposition

Every smooth map $Y \to X$ factors locally as

$$Y \xrightarrow{f} \mathbb{A}^n_X \to X$$

where $\mathbb{A}^n_X := \mathbb{A}^n \times_k X$ and f is étale.

Étale Covers

Loosely, étale morphisms are the algebro-geometric analogues of local homeomorphisms in differential geometry.

You can even build *covering spaces* out of them! An **étale cover** is a surjective étale map $f : Y \to X$.

There's even an **étale fundamental group** $\pi_1(X, x)$ that acts very much like the topological fundamental group of a manifold.

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Example

When $X = \operatorname{Spec} k$, étale covers are $\coprod \operatorname{Spec} L_i \to \operatorname{Spec} k$ where L_i/k are finite separable extensions. Hence $\pi_1(\operatorname{Spec} k) = \operatorname{Gal}(k^{sep}/k)$.

Étale Descent

Returning to our moduli problem context, recall:

Definition

A **stack** is a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ satisfying *descent*: for any étale cover* $\{U_i \to T\}$, the objects/morphisms of $\mathcal{X}(T)$ correspond bijectively to compatible objects/morphisms of $\{\mathcal{X}(U_i)\}$.

Étale Descent

Definition

A **stack** is a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ satisfying *descent*.

More rigorously, for an affine scheme T and a collection of étale maps $\{U_i \to T\}$ such that $\coprod U_i \to T$ is surjective, define a category $\mathcal{X}(\{U_i \to T\})$ with:

- objects: $(\{u_i\}, \{\sigma_{ij}\})$ where $u_i \in \mathcal{X}(U_i)$ and $\sigma_{ij} : u_i \xrightarrow{\sim} u_j$ satisfying $\sigma_{ik} = \sigma_{jk}\sigma_{ij}$ for all i, j, k
- morphisms: $\{u_i' \to u_i\}$ commuting with the σ_{ij}.

We say $\mathcal{X}$ satisfies **étale descent** if for every such $\{U_i \to T\}$, the natural morphism $\mathcal{X}(T) \to \mathcal{X}(\{U_i \to T\})$ is an equivalence of categories (groupoids).

Étale Descent

Definition

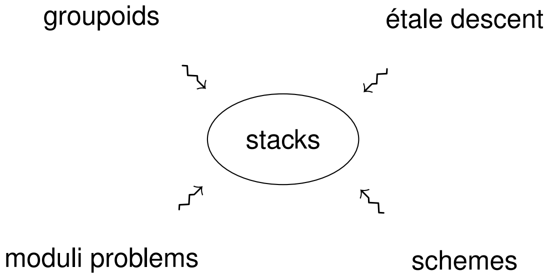
A **stack** is a functor $\mathcal{X} : \texttt{AffSch}^{op} \to \texttt{Gpd}$ satisfying *descent*.

Takeaway: stacks are groupoid-valued functors (usually moduli problems) that have a local-global principle: data on étale "neighborhoods" $U_i \to T$ glues together to give data on all of T.

Summary

Definition

A **stack** is a functor $\mathcal{X} : \texttt{AffSch}^{op} \to \texttt{Gpd}$ satisfying *descent*.



Representability

Important Fact: For every scheme X, its functor of points

$$X(-) : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$$

is a stack. (Can you see why?)

Remember that $X(T) = \mathrm{Hom}(T, X)$.

Representability

Important Fact: For every scheme X, $X(-) = \mathrm{Hom}(-, X) : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ is a stack.

Definition

A functor $\mathcal{X}$ is **representable** if there is a natural isomorphism $\mathcal{X} \cong X(-)$ for a scheme X.

So a moduli problem $\mathscr{M}$ is *representable* when $\mathscr{M} \cong \mathrm{Hom}(-, M)$ for a scheme M, called a *fine moduli space*.

If $\mathscr{M}$ is not representable by a scheme, we can still ask for it to be (represented by) a stack.

Example

For a group G, $\mathrm{Bun}_G(-) : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ is represented by a stack: $\mathrm{Bun}_G(-) \cong \mathrm{Hom}(-, BG)$.

G-Bundles Revisited

Example

For a group G, let $\mathrm{Bun}_G(-) : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ be the moduli problem

$$\mathrm{Bun}_G(T) = \text{the groupoid of principal } G\text{-bundles over } T.$$

(These are sometimes called G-*torsors* in algebraic geometry.)

Fact: $\mathrm{Bun}_G(-)$ is a stack but is not (representable by) a scheme.

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Fact: $\mathrm{Bun}_G(-)$ is a stack but is not (representable by) a scheme.

Proof: $\mathrm{Bun}_G(-) \cong \mathrm{Hom}(-, BG)$. However, it is not representable by a scheme, by a similar proof to the elliptic curves example.

Elliptic Curves Revisited

Let $\mathscr{M}_{1,1} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ be the moduli problem of elliptic curves,

$$\mathscr{M}_{1,1}(T) = \text{groupoid of morphisms } C \to T, \text{ fibres} = \text{ell. curves.}$$

We saw that $\mathscr{M}_{1,1}$ is not represented by a scheme (well, a cx mfld but the same proof works). However:

Theorem

$\mathscr{M}_{1,1} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ *is a stack.*

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Proof.

- Let $\{U_i \to T\}$ be an étale covering.

$\square$

Elliptic Curves Revisited

Theorem

$\mathscr{M}_{1,1} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ *is a stack.*

Proof.

- Let $\{U_i \to T\}$ be an étale covering.
- The functor $\mathscr{M}_{1,1}(T) \to \mathscr{M}_{1,1}(\{U_i \to T\})$ sends an elliptic curve $E \to T$ to the data $(\{E_i\}, \{\sigma_{ij}\})$ where $E_i = E \times_T U_i \to U_i$ and $\sigma_{ij} : E_i \xrightarrow{\sim} E_j$ are compatible isomorphisms, i.e. $\sigma_{ik} = \sigma_{jk}\sigma_{ij}$.

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- This is an equivalence of categories because for any such data $(\{E_i\}, \{\sigma_{ij}\})$, the σ_{ij} allow us to glue the E_i into a scheme $E \to T$; one can check E is elliptic.

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- (Morphisms are similar: local data glues together along the σ_{ij}.)
- Therefore $\mathscr{M}_{1,1}$ satisfies étale descent.

□

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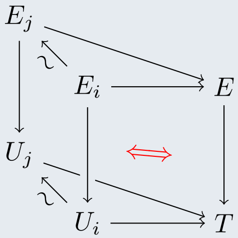
Proof.

Or:



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Theorem

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More interesting: $\mathscr{M}_{1,1}$ is an *algebraic stack*, meaning it admits a smooth surjection $X \to \mathscr{M}_{1,1}$ from a scheme X.

Recall that every elliptic curve admits a Weierstrass equation

$$y^2 = x^3 + Ax + B.$$

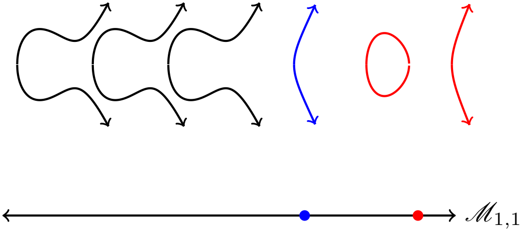
Let $X = \operatorname{Spec} k[x, y, A, B]$. To get a map $X \to \mathscr{M}_{1,1}$, it is equivalent to specify an elliptic curve over X:

$$E = \operatorname{Spec}\big(k[x, y, A, B]/(y^2 - x^3 - Ax - B)\big) \longrightarrow X.$$

One can check that, away from $\Delta := -16(4A^3 + 27B^2) = 0$, $X \to \mathscr{M}_{1,1}$ is a smooth surjection.

Elliptic Curves Revisited

Next: let's explore the geometry of the stack $\mathscr{M}_{1,1}$.



The Point(s) of Stacks

In set theory, a point is rigid: it's just a one-element set $\{x\}$.

In topology, a point could be: a set-theoretic point OR a contractible space (a pt. up to homotopy).

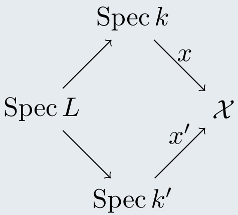
In algebraic geometry, points come in many different flavors:

- For any field k, the underlying space of $\operatorname{Spec} k$ is a point. Field extensions $L/k \longleftrightarrow$ nontrivial maps of points $\operatorname{Spec} L \to \operatorname{Spec} k$.
- (Nilpotents are invisible) The underlying space of $\operatorname{Spec} k[x]/(x^2)$ is also a point, but $k[x]/(x^2) \not\cong k$ so the respective schemes are distinct.
- (Points can be dense) If A is an integral domain with fraction field K, the image of the corresponding map $\operatorname{Spec} K \hookrightarrow \operatorname{Spec} A$ is called the *generic point* of $\operatorname{Spec} A$. It is dense in the topology on $\operatorname{Spec} A$.

The Point(s) of Stacks

Definition

A **point** of a scheme X is an equivalence class of morphisms $x : \operatorname{Spec} k \to X$, where k is a field, and where two points $x : \operatorname{Spec} k \to X$ and $x' : \operatorname{Spec} k' \to X$ are equivalent if there exists a field $L \supseteq k, k'$ making the following commute:



The Point(s) of Stacks

Replacing the scheme X with a stack $\mathcal{X}$ yields the same definition.

Definition

The **automorphism group** of a point $x : \operatorname{Spec} k \to \mathcal{X}$ of a stack $\mathcal{X}$ is the pullback $\operatorname{Aut}(x)$ in the diagram

$$\begin{array}{ccc} \operatorname{Aut}(x) & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow{\scriptstyle (x,x)} \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times \mathcal{X} \end{array}$$

A **stacky point** of $\mathcal{X}$ is a point with $\operatorname{Aut}(x) \neq 1$.

The Moduli Stack $\mathscr{M}_{1,1}$

Proposition

Let E be an elliptic curve over $\mathbb{C}$. Then $\mathrm{Aut}(E)$ is a finite cyclic group, and more specifically,

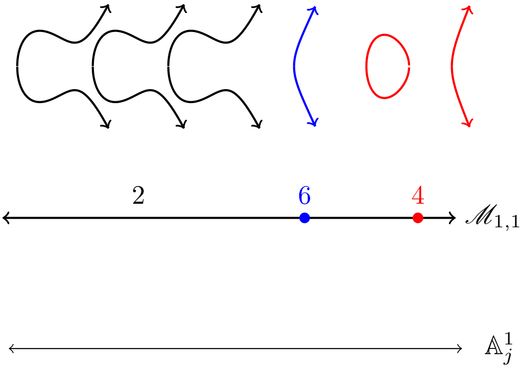
$$\mathrm{Aut}(E) = \begin{cases} \mathbb{Z}/6\mathbb{Z}, & \text{if } j(E) = 0 \\ \mathbb{Z}/4\mathbb{Z}, & \text{if } j(E) = 1728 = 12^3 \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } j(E) \neq 0, 1728. \end{cases}$$

Theorem

Let $\mathscr{M}_{1,1}$ be the moduli stack of elliptic curves. Then $\mathscr{M}_{1,1}$ is a "stacky" $\mathbb{A}^1$ with automorphism group $\mathbb{Z}/2\mathbb{Z}$ at every point except two, which have automorphism groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$. Further, there is a map $\mathscr{M}_{1,1} \to \mathbb{A}^1_j$ which is a bijection on geometric points and is universal with respect to maps $\mathscr{M}_{1,1} \to M$ where M is a scheme.

The Moduli Stack $\mathscr{M}_{1,1}$

That is, $\mathscr{M}_{1,1}$ looks like



The Moduli Stack $\mathscr{M}_{1,1}$

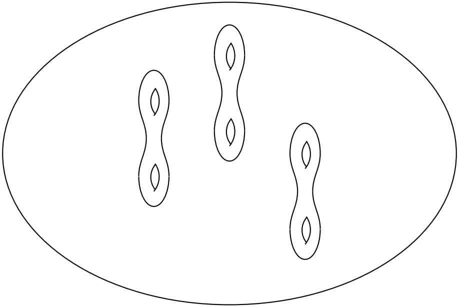
Utility of $\mathscr{M}_{1,1}$ over $\mathbb{A}^1_j$:

- Stores info about automorphisms that may even be invisible over subfields of $\mathbb{C}$, e.g. $\mathbb{Q}$.
- Classical objects (e.g. modular forms) have geometric interpretation over a nice compactification $\overline{\mathscr{M}}_{1,1}$ (e.g. as sections of line bundles), vs. ad hoc and awkward interpretation over $\mathbb{P}^1_j = \overline{\mathbb{A}^1_j}$.
- Counting formulas with fractional coefficients "come from geometry".
- Accommodates extra structure like level structure (elliptic curves with torsion), polarizations, etc.
- "Lifts to topology" as a spectral (Deligne–Mumford) stack, leading to topological modular forms.

Higher Genus Curves

All of this generalizes to curves of genus $g \geq 2$:

Let $\mathscr{M}_g$ be the moduli functor sending $T \mapsto$ the groupoid of families of curves $C \to T$ with genus g fibres.



Over $\mathbb{C}$, this is equivalent to the groupoid of families of Riemann surfaces of genus g.

Higher Genus Curves

Quick facts about $\mathscr{M}_g$ for $g \geq 2$:

- $\mathscr{M}_g$ is a (Deligne–Mumford) stack that is, once again, not representable by a scheme.
- But it has a coarse moduli space M_g which has been well-studied.
- $\dim \mathscr{M}_g = \dim M_g = 3g - 3$ (compare to hyperbolic geometry).
- Marked version has deep ties to number theory, e.g. $\chi(\mathscr{M}_{g,1}) = \zeta(1 - 2g)$.

Higher Genus Curves

In fact, the formula $\chi(\mathscr{M}_{g,1}) = \zeta(1-2g)$ (due to Harer–Zagier) holds even for $g = 1$:

$$\chi(\mathscr{M}_{1,1}) = \zeta(-1) = \frac{1}{12}.$$

Here, we are using *orbifold Euler characteristic*:

$$\chi(\mathscr{M}) = |V| - |E| + |F| = \sum_{v \in V} \frac{1}{|\operatorname{Aut}(v)|} - \sum_{e \in E} \frac{1}{|\operatorname{Aut}(e)|} + |F|$$

Thank you!

https://www.desmos.com/calculator/ialhd71we3