

Pretalk: Objective Linear Algebra

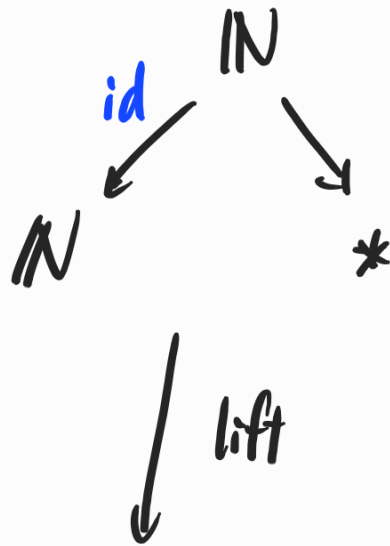
In the main talk, we'll see how to lift zeta functions

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{the Riemann zeta function}$$

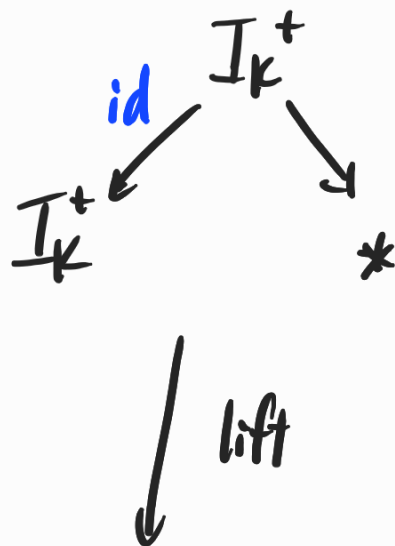
$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s} \quad \text{for a number field } K/\mathbb{Q}$$

$$\begin{aligned} Z(X, t) &= \exp \left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_q^n)}{n} t^n \right] \quad \text{for a variety } X/\mathbb{F}_q \\ &= \sum_{\alpha \in Z_0^{\text{eff}}(X)} 1 t^{\deg(\alpha)} \end{aligned}$$

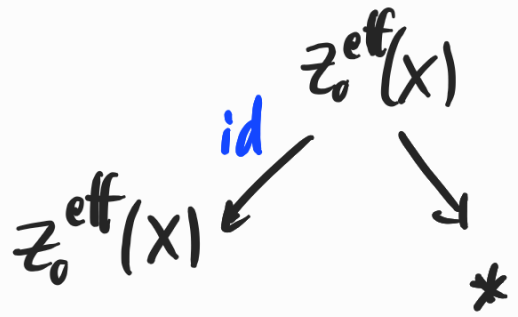
to an "objective incidence algebra":



$$\Psi_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \longleftrightarrow (1, 1, 1, \dots) \in I(\mathbb{Q})$$



$$\Psi_K(s) = \sum_{d \in \mathcal{O}_K} \frac{1}{N(d)^s} \longleftrightarrow (1, 1, 1, \dots) \in I(K)$$



$$Z(X, t) = \sum_{\alpha \in Z_0^{\text{eff}}(X)} 1 t^{\deg(\alpha)} \longleftrightarrow (1, 1, 1, \dots) \in I(X)$$

The key idea is that

sets categorify numbers.

Objective linear algebra (due to Lawvere,
Menni, et al.): linear algebra with sets.

If the only scalars we're interested in are $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, we can lift many concepts from linear algebra to the category **Set** and recover original versions by taking cardinality:

Numerical

scalar a

$+$, \times

ground field k

basis B

vector v in the
basis B

scalar multiplication

$a \cdot v$

Objective

set A

\sqcup , \times

category **Set**

set B

set map $v: V \rightarrow B$

product map

$A \times V \rightarrow V \rightarrow B$

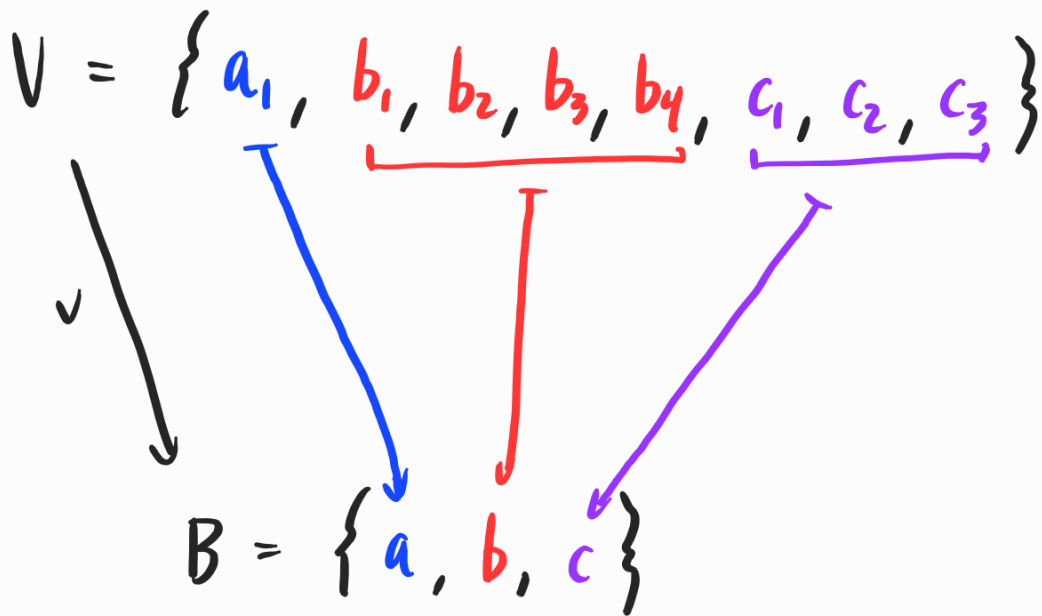
addition $v+w$

sum $V \oplus W \rightarrow B$

vector space kB

slice category Set/B

Ex The vector $v = (1, 4, 3)$ can be encoded by the objective vector



Next: an objective matrix is a span

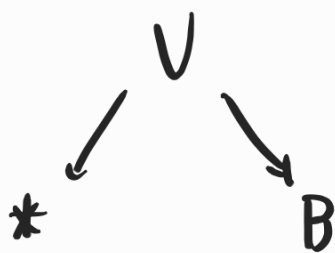


B ↙ ↘ C

Think: a row of column vectors

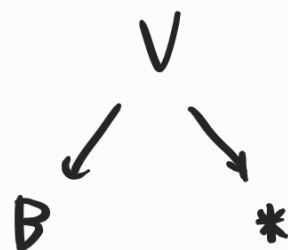
OR a column of row vectors!

A vector $v: V \rightarrow B$ can be viewed
as



a column vector

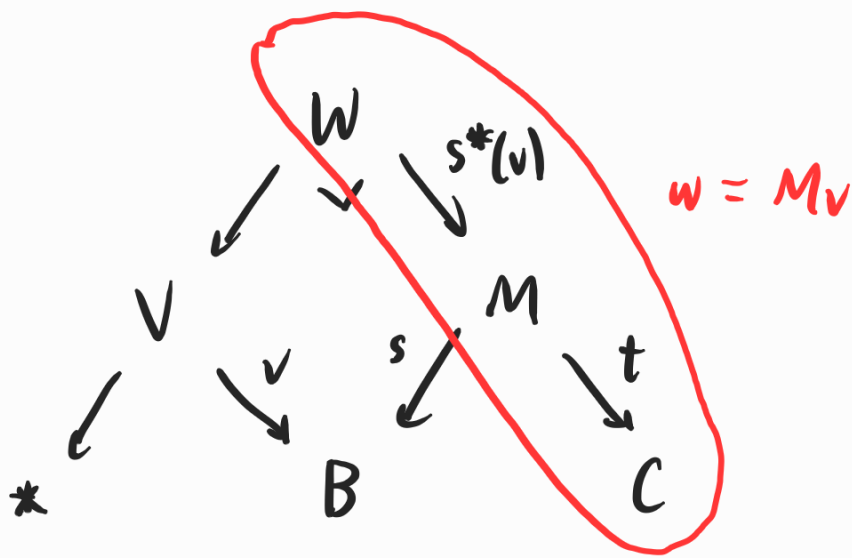
OR



a row vector

To multiply a matrix by a vector,

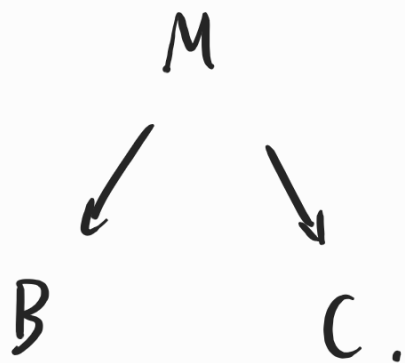
take the pullback:



Exercise: Represent the matrix

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

as a span



A matrix M induces a linear

$$B \xrightarrow{u} C$$

functor $t! s^* : \text{Set}/B \rightarrow \text{Set}/C$ where

$$s^* : \text{Set}/B \rightarrow \text{Set}/M$$

$$\left(\begin{array}{c} V \\ \downarrow \\ B \end{array} \right) \mapsto \left(\begin{array}{ccc} & s^*V & \\ \swarrow & \downarrow & \searrow \\ V & & M \\ \downarrow & & \swarrow \\ & B & \end{array} \right)$$

$$t! : \text{Set}/M \rightarrow \text{Set}/C$$

$$\left(\begin{array}{c} X \\ \downarrow \\ M \end{array} \right) \mapsto \left(\begin{array}{ccc} X & & \\ \downarrow & \dashrightarrow & \\ M & \rightarrow & C \end{array} \right)$$

This gives us a category LIN with

objects = slice categories Set/B

morphisms = linear functors $\text{Set}/B \rightarrow \text{Set}/C$

More concepts that lift to OLA:

$$V \otimes W \rightsquigarrow \text{Set}/_B \otimes \text{Set}/_C := \text{Set}/_{B \times C}$$

$$\text{Hom}(V, W) \rightsquigarrow \text{LIN}(\text{Set}/_B, \text{Set}/_C)$$

$$V^* \rightsquigarrow \text{LIN}(\text{Set}/_B, \text{Set})$$

Q: What if we need scalars other than \mathbb{N}_0 ?

Main talk: L-functions have complex coefficients in general.

Solution: replace the category Set with
a (symmetric monoidal) category of
spaces \mathcal{S} , leading to **homotopy linear
algebra** (Gálvez-Carrillo, Kock, Tanke).

Key examples:

$\mathcal{S} = \text{Grp-Rep} \rightsquigarrow$ L-functions

$\mathcal{S} = \text{Top} \rightsquigarrow$ Macdonald polynomials, etc.

$\mathcal{S} = \text{Var}_k \rightsquigarrow$ motivic zeta functions

$\mathcal{S} = \text{Stack}_k \rightsquigarrow$ motivic L-functions
+ more?

\mathcal{S} = locally compact
abelian groups

\rightsquigarrow archimedean factors?

