Pretalk: Objective Linear Algebra

In the main talk, we'll see how to lift zeta functions

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

the Riemann zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}$$

for a number field $K/\mathbb{Q}$

$$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$$

for a variety $X/\mathbb{F}_q$

$$= \sum_{\alpha \in Z_0^{\mathrm{eff}}(X)} 1\, t^{\deg(\alpha)}$$

to an "objective incidence algebra":

$$\begin{tikzcd} & \mathbb{N} \arrow[dl, "\mathrm{id}"'] \arrow[dr] & \\ \mathbb{N} & & * \end{tikzcd}$$

$$\Big\downarrow \text{lift}$$

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \longleftrightarrow (1, 1, 1, \ldots) \in I(\mathbb{Q})$$

$$\begin{tikzcd} & I_K^+ \arrow[dl, "\mathrm{id}"'] \arrow[dr] & \\ I_K^+ & & * \end{tikzcd}$$

$$\Big\downarrow \text{lift}$$

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s} \longleftrightarrow (1, 1, 1, \ldots) \in I(K)$$

$$\begin{tikzcd} & Z_0^{\mathrm{eff}}(X) \arrow[dl, "\mathrm{id}"'] \arrow[dr] & \\ Z_0^{\mathrm{eff}}(X) & & * \end{tikzcd}$$

$$\Big\downarrow \text{lift}$$

$$Z(X,t) = \sum_{\alpha \in Z_0^{\mathrm{eff}}(X)} 1\, t^{\deg(\alpha)} \longleftrightarrow (1, 1, 1, \ldots) \in I(X)$$

The key idea is that

sets categorify numbers.

Objective linear algebra (due to Lawvere, Menni, et al.): linear algebra with sets.

If the only scalars we're interested in are $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, we can lift many concepts from linear algebra to the category Set and recover original versions by taking cardinality:

Numerical	Objective
scalar a	set A
$+, \times$	$\amalg, \times$
ground field k	category Set
basis B	set B
vector v in the basis B	set map $v : V \to B$
scalar multiplication av	product map $A \times V \to V \to B$

$A \times V \to V \to B$

addition $v + w$ — sum $V \amalg W \to B$

vector space kB — slice category $\mathrm{Set}/_B$

$\boxed{\text{Ex}}$ The vector $v = (1, 4, 3)$ can be encoded by the objective vector

$$\begin{tikzcd} V = \{a_1, b_1, b_2, b_3, b_4, c_1, c_2, c_3\} \arrow[d, "v"'] \\ B = \{a, b, c\} \end{tikzcd}$$

Next: an objective matrix is a span

$$\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr, "t"] & \\ {} & & {} \end{tikzcd}$$

$$B \qquad\qquad C$$

Think: a row of column vectors

OR a column of row vectors!

A vector $v : V \longrightarrow B$ can be viewed as

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ * & & B \end{tikzcd} \qquad \text{OR} \qquad \begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ B & & * \end{tikzcd}$$

a column vector

a row vector

To multiply a matrix by a vector, take the pullback:

$$\begin{tikzcd}
 & & W \arrow[dl] \arrow[dr, "s^*(v)"] & & \\
 & V \arrow[dl] \arrow[dr, "v"] & & M \arrow[dl, "s"'] \arrow[dr, "t"] & \\
* & & B & & C
\end{tikzcd}$$

$w = Mv$

Exercise: Represent the matrix

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

as a span

$$\begin{tikzcd}
 & M \arrow[dl] \arrow[dr] & \\
B & & C.
\end{tikzcd}$$

A matrix $\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr, "t"] & \\ & & \end{tikzcd}$ induces a linear

$B \qquad\qquad C$

functor $t_! s^* : \mathrm{Set}_{/B} \longrightarrow \mathrm{Set}_{/C}$ where

$s^* : \mathrm{Set}_{/B} \longrightarrow \mathrm{Set}_{/M}$

$$\left(\begin{tikzcd} V \arrow[d] \\ B \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} & s^*V \arrow[dl] \arrow[dr] \arrow[dd, phantom, "\lrcorner" very near start] & \\ V \arrow[dr] & & M \arrow[dl, "s"] \\ & B & \end{tikzcd}\right)$$

$t_! : \mathrm{Set}_{/M} \longrightarrow \mathrm{Set}_{/C}$

$$\left(\begin{tikzcd} X \arrow[d] \\ M \end{tikzcd}\right) \longmapsto \left(\begin{tikzcd} X \arrow[d] \arrow[dr, dashed] & \\ M \arrow[r] & C \end{tikzcd}\right)$$

This gives us a category LIN with

objects = slice categories $\mathrm{Set}_{/B}$

morphisms = linear functors $\mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$

More concepts that lift to OLA:

$$V \otimes W \rightsquigarrow \mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} := \mathrm{Set}_{/B \times C}$$

$$\mathrm{Hom}(V, W) \rightsquigarrow \mathrm{LIN}(\mathrm{Set}_{/B}, \mathrm{Set}_{/C})$$

$$V^* \rightsquigarrow \mathrm{LIN}(\mathrm{Set}_{/B}, \mathrm{Set})$$

Q: What if we need scalars other than $\mathbb{N}_0$?

Main talk: L-functions have complex coefficients in general.

Solution: replace the category Set with a (symmetric monoidal) category of spaces $\mathcal{S}$, leading to **homotopy linear algebra** (Gálvez-Carrillo, Kock, Tonks).

Key examples:

$\mathcal{S} = G_{\mathbb{Q}}\text{-Rep} \rightsquigarrow$ L-functions

$\mathcal{S} = \mathrm{Top} \rightsquigarrow$ Macdonald polynomials, etc.

$\mathcal{S} = \mathrm{Var}_k \rightsquigarrow$ motivic zeta functions

$\mathcal{S} = \mathrm{Stack}_k \rightsquigarrow$ motivic L-functions

+ more ?

$\int$ = locally compact abelian groups $\rightsquigarrow$ archimedean factors?