Categorifying zeta and L-functions

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Number Theory Seminar

May 18, 2023

Joint work with Jon Aycock

Introduction

Believe women.

Believe your colleagues.

Stop the cruelty.

Introduction

Based on

Categorifying quadratic zeta functions

Jon Aycock, Andrew Kobin

The Dedekind zeta function of a quadratic number field factors as a product of the Riemann zeta function and the L-function of a quadratic Dirichlet character. We categorify this formula using objective linear algebra in the abstract incidence algebra of the division poset.

Comments:	27 pages
Subjects:	**Number Theory (math.NT)**
MSC classes:	11M06, 11M41, 18N50, 06A11, 16T10
Cite as:	arXiv:2205.06298 **[math.NT]**
	(or arXiv:2205.06298v1 **[math.NT]** for this version)

and

Categorifying Zeta Functions of Hyperelliptic Curves

Jon Aycock, Andrew Kobin

The zeta function of a hyperelliptic curve C over a finite field factors into a product of L-functions, one of which is the L-function of C. We categorify this formula using objective linear algebra in the abstract incidence algebra of the poset of effective 0-cycles of C. As an application, we prove a collection of combinatorial formulas relating the number of ramified, split and inert points on C to the overall point count of C.

Comments:	20 pages
Subjects:	**Number Theory (math.NT)**; Algebraic Geometry (math.AG)
MSC classes:	14G10, 11G20, 18N50, 06A11, 16T10
Cite as:	arXiv:2304.13111 **[math.NT]**
	(or arXiv:2304.13111v1 **[math.NT]** for this version)

as well as work in progress with Jon Aycock.

Introduction

Motivation: How are different zeta and L-functions related? Do they fit into a common framework?

motivic L-functions

$$Z_{mot}(X,t)$$

arithmetic L-functions

$$\zeta_{\mathbb{Q}}(s), \zeta_K(s), \zeta_X(s)$$

local L-functions

$$Z(X/\mathbb{F}_q, t)$$

Introduction

Motivation: How are different zeta and L-functions related? Do they fit into a common framework?

$$\begin{tikzcd}
 & \begin{array}{c}\text{motivic } L\text{-functions}\\ Z_{mot}(X,t)\end{array} \arrow[dr, "\text{motivic measure } \#_q"] & \\
\begin{array}{c}\text{arithmetic } L\text{-functions}\\ \zeta_{\mathbb{Q}}(s), \zeta_K(s), \zeta_X(s)\end{array} & & \begin{array}{c}\text{local } L\text{-functions}\\ Z(X/\mathbb{F}_q, t)\end{array} \arrow[ll, hook', "\text{local factors}"']
\end{tikzcd}$$

Arithmetic Functions

A good starting place is always with the Riemann zeta function:

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Arithmetic Functions

This is an example of a Dirichlet series:

$$F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}.$$

We will focus on the formal properties of Dirichlet series.

The coefficients $f(n)$ assemble into an **arithmetic function** $f : \mathbb{N} \to \mathbb{C}$. (Think: F is a generating function for f.)

Then $\zeta_{\mathbb{Q}}(s)$ is the Dirichlet series for $\zeta : n \mapsto 1$.

Arithmetic Functions

The space of arithmetic functions $A = \{f : \mathbb{N} \to \mathbb{C}\}$ form an algebra under convolution:

$$(f * g)(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).$$

This identifies the algebra of formal Dirichlet series with A:

$$\begin{aligned}
A &\longleftrightarrow DS(\mathbb{Q}) \\
f &\longmapsto F(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s} \\
f * g &\longmapsto F(s)G(s) \\
\zeta &\longmapsto \zeta_{\mathbb{Q}}(s)
\end{aligned}$$

Arithmetic Functions over Number Fields

For a number field $K/\mathbb{Q}$, its zeta function can be written

$$\zeta_K(s) = \sum_{\mathfrak{a} \in I_K^+} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{\#\{\mathfrak{a} \mid N(\mathfrak{a}) = n\}}{n^s}$$

where $I_K^+ = \{\text{ideals in } \mathcal{O}_K\}$ and $N = N_{K/\mathbb{Q}}$.

Arithmetic Functions over Number Fields

As with $\zeta_{\mathbb{Q}}(s)$, we can formalize certain properties of $\zeta_K(s)$ in the algebra of arithmetic functions $A_K = \{f : I_K^+ \to \mathbb{C}\}$ with

$$(f * g)(\mathfrak{a}) = \sum_{\mathfrak{b}|\mathfrak{a}} f(\mathfrak{b})g(\mathfrak{a}\mathfrak{b}^{-1}).$$

This admits an algebra map to $DS(\mathbb{Q})$:

$$\begin{aligned} N_* : A_K &\longrightarrow A \cong DS(\mathbb{Q}) \\ f &\longmapsto \left(N_* f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right) \\ \zeta &\longmapsto N_*\zeta \leftrightarrow \zeta_K(s) \end{aligned}$$

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Interpretation: N allows us to build Dirichlet series for arithmetic functions over K.

Varieties over Finite Fields

Let X be an algebraic variety over $\mathbb{F}_q$. Its point-counting zeta function is the power series

$$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$$

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$$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$$

Once again, we can formalize certain properties of $Z(X,t)$ in an algebra of arithmetic functions.

Let $Z_0^{\text{eff}}(X)$ be the set of effective 0-cycles on X, i.e. formal $\mathbb{N}_0$-linear combinations of closed points of X, written $\alpha = \sum m_x x$.

Let $A_X = \{f : Z_0^{\text{eff}}(X) \to \mathbb{C}\}$ be the algebra of arithmetic functions with

$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta) g(\alpha - \beta).$$

We call the distinguished element $\zeta : \alpha \mapsto 1$ the *zeta function* of X.

Varieties over Finite Fields

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Varieties over Finite Fields

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$$(f * g)(\alpha) = \sum_{\beta \leq \alpha} f(\beta) g(\alpha - \beta).$$

This time, there's no map to $DS(\mathbb{Q})$... but there's a map to the algebra of formal power series:

$$A_X \longrightarrow A_{\operatorname{Spec} \mathbb{F}_q} \cong \mathbb{C}[[t]]$$

$$f \leftrightarrow \sum_{n=0}^{\infty} f(n) t^n$$

$$f \longmapsto \text{“}\deg_*(f)\text{”}$$

$$\zeta \longmapsto \text{“}\deg_*(\zeta)\text{”} \leftrightarrow Z(X, t)$$

What's really going on?

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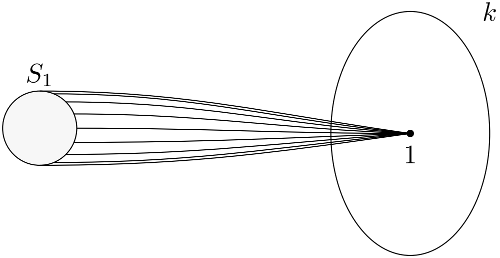
A, A_K and A_X are examples of **reduced incidence algebras**, which come from a much more general simplicial framework.



Numerical Incidence Algebras

Idea (due to Gálvez-Carrillo, Kock and Tonks): zeta functions come from higher homotopy structure.

In this talk: zeta functions come from decomposition sets.

In general: zeta functions come from decomposition spaces.

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \leftleftarrows S_1 \cdots S_2 \cdots .$$

Example

A category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ with:

- 0-simplices $=$ objects x in $\mathcal{C}$
- 1-simplices $=$ morphisms $x \xrightarrow{f} y$ in $\mathcal{C}$
- 2-simplices $=$ decompositions $x \xrightarrow{h} y = x \xrightarrow{f} z \xrightarrow{g} y$
- etc.

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \rightleftarrows S_1 \; \substack{\leftarrow\\\rightarrow\\\leftarrow\\\rightarrow\\\leftarrow} \; S_2 \cdots .$$

Example

For a number field $K/\mathbb{Q}$, I_K^+ is a simplicial set with:

- 0-simplices $=$ ideals $\mathfrak{a}$ in $\mathcal{O}_K$
- 1-simplices $=$ divisibility $\mathfrak{b} \to \mathfrak{a} \iff \mathfrak{a} \mid \mathfrak{b}$
- 2-simplices $=$ decompositions $\mathfrak{b} \to \mathfrak{c} \to \mathfrak{a}$
- etc.

Numerical Incidence Algebras

Recall: a **simplicial set** is a functor $S : \Delta^{op} \to \mathrm{Set}$

$$S_0 \rightleftarrows S_1 \rightleftarrows S_2 \cdots .$$

Example

For a variety $X/\mathbb{F}_q$, $Z_0^{\mathrm{eff}}(X)$ is a simplicial set with:

- 0-simplices $=$ effective 0-cycles α
- 1-simplices $=$ relations $\alpha \leq \beta$
- 2-simplices $=$ decompositions $\alpha \leq \gamma \leq \beta$
- etc.

Numerical Incidence Algebras

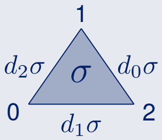
A certain type of simplicial set called a **decomposition set** defined by Gálvez-Carrillo, Kock and Tonks admits a notion of incidence algebra.

Definition

The **numerical incidence algebra** of a decomposition set S is the vector space $I(S) = \mathrm{Hom}(k[S_1], k)$ with multiplication

$$I(S) \otimes I(S) \longrightarrow I(S)$$

$$f \otimes g \longmapsto (f * g)(x) = \sum_{\substack{\sigma \in S_2 \\ d_1\sigma = x}} f(d_2\sigma)g(d_0\sigma).$$



Numerical Incidence Algebras

In $I(S) = \mathrm{Hom}(k[S_1], k)$, there is a distinguished element called the **zeta function** $\zeta : x \mapsto 1$.

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Key takeaways:

(1) A zeta function is $\zeta \in I(S)$ for some decomposition set S.

(2) Familiar zeta functions like $\zeta_K(s)$ and $Z(X, t)$ are constructed from some $\zeta \in \widetilde{I}(S)$ by pushing forward to another reduced incidence algebra which can be interpreted in terms of generating functions:

$$\text{e.g.} \quad \widetilde{I}(\mathbb{N}, |) \cong DS(\mathbb{Q}), \qquad \text{e.g.} \quad \widetilde{I}(\mathbb{N}_0, \leq) \cong k[[t]].$$

(3) Some properties of zeta functions can be proven in the incidence algebra directly:

$$\text{e.g.} \quad \zeta_{\mathbb{Q}}(s) = \prod_p \frac{1}{1 - p^{-s}} \longleftrightarrow \widetilde{I}(\mathbb{N}, |) \cong \bigotimes_p \widetilde{I}(\{p^k\}, |).$$

Numerical Incidence Algebras

Okay, so far: $\zeta_{\mathbb{Q}}(s), \zeta_K(s), Z(X,t)$, etc. lift to the same framework.

Next: how can we get them talking to each other?

Objective Linear Algebra

The construction of $I(S)$ can be generalized further using the formalism of **objective linear algebra** (“linear algebra with sets”):

Numerical	Objective
basis B	set B
vector v	set map $v : X \to B$
matrix M	span $B \xleftarrow{s} M \xrightarrow{t} C$
vector space V	slice category $\mathrm{Set}_{/B}$
linear map with matrix M	linear functor $t_! s^* : \mathrm{Set}_{/B} \to \mathrm{Set}_{/C}$
tensor product $V \otimes W$	$\mathrm{Set}_{/B} \otimes \mathrm{Set}_{/C} \cong \mathrm{Set}_{/B \times C}$

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$\longleftarrow$

To recover vector spaces, take $V = k^B$ and take cardinalities.

Abstract Incidence Algebras

How do we construct $I(S)$ as an “objective vector space”?

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So an element $f \in I(S)$ is a linear functor $f = t_! s^* : \mathrm{Set}_{/S_1} \to \mathrm{Set}$ represented by a span

$$f = \left(\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr, "t"] & \\ S_1 & & * \end{tikzcd} \right)$$

Abstract Incidence Algebras

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Example

The **zeta functor** is the element $\zeta \in I(S)$ represented by

$$\zeta = \left(\begin{tikzcd} & S_1 \arrow[dl, "\mathrm{id}"'] \arrow[dr] & \\ S_1 & & * \end{tikzcd} \right)$$

Abstract Incidence Algebras

Example

For two elements $f, g \in I(S)$ represented by

$$f = \left(\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr] & \\ S_1 & & * \end{tikzcd} \right) \quad \text{and} \quad g = \left(\begin{tikzcd} & N \arrow[dl, "t"'] \arrow[dr] & \\ S_1 & & * \end{tikzcd} \right)$$

the convolution $f * g \in I(S)$ is represented by

$$(f * g) = \left(\begin{tikzcd} & & P \arrow[dl] \arrow[dr] & & \\ & S_2 \arrow[dl, "d_1"'] \arrow[dr, "(d_2, d_0)"] & & M \times N \arrow[dl, "s \times t"'] \arrow[dr] & \\ S_1 & & S_1 \times S_1 & & * \end{tikzcd} \right)$$

Abstract Incidence Algebras

Advantages of the objective approach:

- Intrinsic: zeta is built into the object S directly
- Functorial: to compare zeta functions, find the right map $S \to T$
- Structural: proofs are categorical, avoiding choosing elements (e.g. computing local factors of zeta functions explicitly is difficult)
- It's pretty fun to prove things!

Quadratic Zeta Functions

For a quadratic number field $K/\mathbb{Q}$, the zeta function $\zeta_K(s)$ satisfies

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s)L(\chi, s)$$

where $L(\chi, s)$ is the L-function attached to the Dirichlet character $\chi = \left(\frac{D}{\cdot}\right)$, where $D =$ disc. of K.

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where $L(\chi, s)$ is the L-function attached to the Dirichlet character $\chi = \left(\frac{D}{\cdot}\right)$, where $D =$ disc. of K.

Theorem (Aycock–K.,'22)

This formula lifts to an equivalence of linear functors in $\widetilde{I}(\mathbb{N}, |)$:

$$N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+$$

where $N : (I_K^+, |) \to (\mathbb{N}, |)$ is the norm and $\chi^+, \chi^- \in I(\mathbb{N}, |)$.

In the numerical incidence algebra, this becomes

$$N_*\zeta_K = \zeta_{\mathbb{Q}} * (\chi^+ - \chi^-) = \zeta_{\mathbb{Q}} * \chi.$$

Elliptic Curves

For an elliptic curve $E/\mathbb{F}_q$, the zeta function $Z(E,t)$ can be written

$$Z(E,t) = \frac{1 - a_q t + qt^2}{(1-t)(1-qt)} = Z(\mathbb{P}^1, t) L(E,t).$$

Theorem (Aycock–K., '23)

In the reduced incidence algebra $\widetilde{I}(Z_0^{\mathrm{eff}}(E))$, there is an equivalence of linear functors

$$\pi_* \zeta_E + \zeta_{\mathbb{P}^1} * L(E)^- \cong \zeta_{\mathbb{P}^1} * L(E)^+$$

where $\pi : E \to \mathbb{P}^1$ is a fixed double cover and $L(E)^+$ and $L(E)^-$ are elements of the incidence algebra $I(Z_0^{\mathrm{eff}}(\mathbb{P}^1))$.

Pushing forward to $\widetilde{I}(Z_0^{\mathrm{eff}}(\operatorname{Spec} \mathbb{F}_q)) \cong k[[t]]$, it already reads

$$t_* \zeta_E = t_* \zeta_{\mathbb{P}^1} * L(E).$$

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

Let $S = (\mathbb{N}, |)$ and $T = (I_K^+, |)$, so that $N : T \to S$ induces

$$N_* : \widetilde{I}(T) \longrightarrow \widetilde{I}(S), \quad f \longmapsto \left(N_* f : n \mapsto \sum_{N(\mathfrak{a})=n} f(\mathfrak{a}) \right).$$

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

Each term in the formula is represented by a span:

$$N_*\zeta_K = \left(\begin{tikzcd} & T_1 \arrow[dl, "N"'] \arrow[dr] & \\ S_1 & & * \end{tikzcd} \right)$$

Sketch of Proof

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Each term in the formula is represented by a span:

$$\zeta_{\mathbb{Q}} * \chi^- = \left(\begin{tikzcd}
 & & P^- \arrow[dl, "\alpha^-"'] \arrow[dr] & & \\
 & S_2 \arrow[dl, "d_1"'] \arrow[dr, "{(d_2, d_0)}"] & & S_1 \times S_1^- \arrow[dl, "id \times j^-"] \arrow[dr] & \\
S_1 & & S_1 \times S_1 & & *
\end{tikzcd}\right)$$

for a certain “vector” $j^- : S_1^- \to S_1$ representing χ^-.

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

Each term in the formula is represented by a span:

$$\zeta_{\mathbb{Q}} * \chi^+ = \left(\begin{tikzcd} & & P^+ \arrow[dl, "\alpha^+"'] \arrow[dr] & & \\ & S_2 \arrow[dl, "d_1"'] \arrow[dr, "(d_2, d_0)"] & & S_1 \times S_1^+ \arrow[dl, "id \times j^+"] \arrow[dr] & \\ S_1 & & S_1 \times S_1 & & * \end{tikzcd} \right)$$

for a certain “vector” $j^+ : S_1^+ \to S_1$ representing χ^+.

Sketch of Proof

$$\boxed{N_*\zeta_K + \zeta_{\mathbb{Q}} * \chi^- \cong \zeta_{\mathbb{Q}} * \chi^+}$$

So the formula is an equivalence of the following spans:

$$\left(\begin{tikzcd} & T_1 \coprod P^- \arrow[dl, "N \sqcup d_1 \circ \alpha^-"'] \arrow[dr] & \\ S_1 & & * \end{tikzcd}\right) \cong \left(\begin{tikzcd} & P^+ \arrow[dl, "d_1 \circ \alpha^+"'] \arrow[dr] & \\ S_1 & & * \end{tikzcd}\right)$$

These are shown to be equivalent prime-by-prime and then assembled into the global formula. $\square$

Higher Order Zeta Functions

More generally, for any Galois extension $K/\mathbb{Q}$, $\zeta_K(s)$ factors into a product of L-functions

$$\zeta_K(s) = \zeta_{\mathbb{Q}}(s) \prod_{\chi \neq 1} L(\chi, s)$$

where χ are the nontrivial irreducible characters of $\mathrm{Gal}(K/\mathbb{Q})$.

Problem: values of $\chi(n)$ land in μ_n in general, so they can't be categorified with sets.

Higher Order Zeta Functions

More generally, for any Galois extension $K/\mathbb{Q}$, $\zeta_K(s)$ factors into a product of L-functions

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where χ are the nontrivial irreducible characters of $\mathrm{Gal}(K/\mathbb{Q})$.

Solution (in progress with J. Aycock): upgrade to simplicial G-representations ($G = G_{\mathbb{Q}}$).

Higher Order Zeta Functions

Actually, let's go for broke: for a(n admissible) G_K-representation V, we define an "L-functor"

$$L(V) = \left(\begin{tikzcd}[column sep=small] & \bigoplus_{n=0}^{\infty} V_n \arrow[dl] \arrow[dr] & \\ \bigoplus_{n=0}^{\infty} R_K & & R_K \end{tikzcd} \right)$$

($R_K =$ modified representation ring incorporating Frobenius actions)

Theorem (Additivity)

For two (admissible) G-representations V, W, there is an equivalence

$$L(V \oplus W) \cong L(V) * L(W)$$

in the incidence algebra $I(\mathbb{Q})$ of L-functors of G-representations.

Higher Order Zeta Functions

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(R_K = modified representation ring incorporating Frobenius actions)

Conjecture (Artin Induction)

For a(n admissible) G_K-representation V, there is an equivalence

$$L\left(\mathrm{Ind}_{G_K}^{G} V\right) \approx N_* L(V)$$

where $N_* : I(K) \to I(\mathbb{Q})$ is the pushforward along the norm map and $\approx$ is "trace equivalence".

Motivic Zeta Functions

For any k-variety X, $Z_{mot}(X,t) = \sum_{n=0}^{\infty} [\mathrm{Sym}^n X] t^n$ decategorifies to other zeta functions by applying motivic measures (point counting, Euler characteristic, etc.)

Das–Howe ('21) lift $Z_{mot}(X,t)$ to a numerical incidence algebra

$$\widetilde{I}_{mot}(\Gamma^{\bullet,+}(X)) = \prod_{n=0}^{\infty} K_0(\mathtt{Var}_{/\Gamma^n X})$$

where $\Gamma^n X$ are the divided powers of X.

Motivic Zeta Functions

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where $\Gamma^n X$ are the divided powers of X.

Work in progress: lift $Z_{mot}(X,t)$ to an objective incidence algebra $I(\Gamma^{\bullet,+}(X))$ in the category of simplicial k-varieties. Passing to K_0 recovers Das and Howe's construction.

More Dreams

Here are some other things we're working on:

- Ihara zeta functions of graphs.
- Study the zeta function of an algebraic stack $\mathcal{X} \to X$ in terms of ζ_X, e.g. over $\mathbb{F}_q$, Behrend defines $Z(\mathcal{X}, t)$ for such a stack.
- Lift motivic L-functions to the objective level and prove formulas, e.g. motivic Artin induction.
- Realize archimedean factors of completed zeta functions as elements of abstract incidence algebras, e.g. the factor at ∞ $\zeta_\infty(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$ lives in a certain Hecke algebra.

Key insight: decomposition sets $\rightsquigarrow$ decomposition spaces

Thank you!