Moduli of Wildly Ramified Covers of Curves

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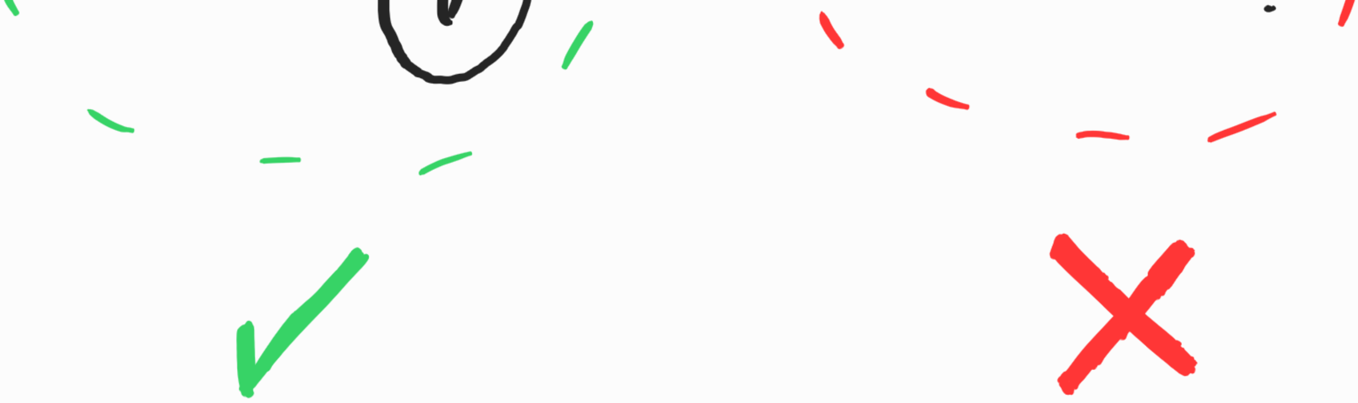
some joint work in progress with V. Thatte

Introduction



The key in many situations is that structures (varieties, schemes, algebraic groups) have a rich topology over $\mathbb{C}$, which they lack over other fields/rings.

In arithmetic geometry, we want to leverage the geometry of these spaces to solve difficult counting problems over $\mathbb{Q}$, $\mathbb{F}_q$, $\mathbb{Q}_p$, $\mathbb{Z}$, $\mathbb{Q}(\sqrt{D})$, $\mathbb{F}_q((t))$, etc.

over $\mathbb{Q}$, $\mathbb{F}_q$, [illegible], etc.

Many ways to replace the usual tools from topology with algebraic tools, including:

(1) topological covers $\rightsquigarrow$ étale covers

flat + unramified

(2) fundamental group $\pi_1^{top}(X)$ $\rightsquigarrow$ étale fundamental group $\pi_1^{ét}(X)$

(3) singular/de Rham cohomology $\rightsquigarrow$ étale/ℓ-adic cohomology

(4) loops, higher homotopy groups, etc. $\rightsquigarrow$???

I'll just focus on (1) and (2) for this intro.

Recall: $\pi_1^{ét}(U)$ is a profinite group whose finite quotients correspond to finite étale covers $V \longrightarrow U$.

Some examples over $\mathbb{C}$:

 $\pi_1^{ét}(\mathbb{P}^1_{\mathbb{C}}) = 1$ (Riemann sphere is simply connected)

 $\pi_1^{ét}(\mathbb{A}^1_{\mathbb{C}}) = 1$ (no unram. covers of affine line)

$$\pi_1^{ét}(\mathbb{A}^1_{\mathbb{C}} \setminus \{0\}) = \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$$

$$\begin{tikzcd} \mathbb{A}^1 \setminus \{0\} \arrow[d] & x \arrow[d, mapsto] \\ \mathbb{A}^1 \setminus \{0\} & x^n \end{tikzcd}$$

In general, if $U = X \setminus \{x_1, \dots, x_n\}$ is a punctured Riemann surface, then

$$\pi_1^{ét}(U) \cong \widehat{\pi_1^{top}(\Sigma_{g,n})}$$

← profin. comp.

genus g sfc. w/ n punct's.

However, over a field k of characteristic

$p > 0$, things are not so nice...

$$\pi_1^{et}(\mathbb{P}^1_k) = 1$$ (the projective line is still "simply connected")

$$\pi_1^{et}(\mathbb{A}^1_k) = \text{¯\\\_(ツ)\_/¯}$$

$\mathbb{Z}/4\mathbb{Z}$ not quasi 5-gp.

By a theorem of Raynaud (1994), we know the finite quotients of $\pi_1^{et}(\mathbb{A}^1_{\bar{k}})$.

G, fix p

$p(G) = \langle P \mid \text{Sylow}_p \rangle$

quasi-p = gen. by Sylow p's

Key idea: étale covers of $\mathbb{A}^1$ come from ramified covers $Y \to \mathbb{P}^1$ that are branched at ∞.

When the isotropy subgroup of $Y \to \mathbb{P}^1$

at a point over ∞ has order divisible by p. we say the cover is wildly ramified (or just wild).

Artin-Schreier Covers of Curves

The basic building blocks of the wild part of $\pi_1^{et}(\mathbb{A}^1)$ are the $\mathbb{Z}/p\mathbb{Z}$ quotients, which come from the Artin-Schreier covers of $\mathbb{P}^1$.

In general, a (ramified) cover of curves

$$Y \longrightarrow \mathbb{P}^1$$

corresponds to a (ramified) field extension

$$k(Y)/k(t) =: L/K.$$

An Artin-Schreier cover is one that induces a Galois extension L/K with $\mathrm{Gal}(L/K) \cong \mathbb{Z}/p\mathbb{Z}$.

Facts: (1) Every AS cover is given by an equation $y^p - y = f(t)$.

(2) The integer $j = \min\{\mathrm{ord}_\infty(f) \,/\, \text{all such } f\}$,

the ramification jump of $Y \to \mathbb{P}^1$,

is an invariant of the cover.

(3) In local coordinates, $Y \to \mathbb{P}^1$

is given by $y^p - y = t^{-j}$

with $p \nmid j$, as long as k is <u>perfect</u>.

(4) The genus of Y (as an alg. curve)

is $g(Y) = \dfrac{(p-1)(j-1)}{2}$.

(5) Work of Pries and others describes

the deformation theory of AS covers

of $\mathbb{P}^1$.

(6) In my thesis, I relate AS covers of curves (including $Y \to X$, $X \neq \mathbb{P}^1$) to the theory of stacky curves by way of Artin-Schreier root stacks.

$\boxed{\text{Ex}}$ Assume k is perfect and consider

$$\begin{array}{c} Y \\ \big\downarrow \\ \mathbb{P}^1 \end{array} : \quad y^p - y = \frac{1}{t^p}$$

$$j = \min\{\operatorname{ord}_\infty(f)\}$$

$$\underbrace{\left(y + \frac{1}{t}\right)^p} - \left(y + \frac{1}{t}\right) = \frac{1}{t^p}$$

$$y^p + \frac{1}{t^p} - y - \frac{1}{t} = \frac{1}{t^p}$$

$$y^p - y = \frac{1}{t^1}$$

$$\boxed{j = 1}$$

$h^p - h$ don't change the iso. of Y

$$f \longmapsto \underline{f + h^p - h}$$

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "\text{ét loc.}"] & \mathbb{P}^1 \arrow[d, "\wp"] \\
\mathbb{P}^1 \arrow[r, "f"] & \mathbb{P}^1
\end{tikzcd}
\qquad
\begin{array}{c} x \\ \downarrow \\ x^p - x \end{array}$$

$\ldots(\ldots) = \deg f^* \mathcal{O}(1)$

$\mathrm{ord}_\infty(f) = \deg + O(1)$

Expanding on (3), if k is not perfect, work of K. Kato and Thatte provide a preferred $f \in K$ for the local equation

$$Y : y^p - y = f(t)$$

(called "best f") whose pole order at ∞ coincides with the Swan conductor of $Y/\mathbb{P}^1$ at ∞.

More generally, if $Y \longrightarrow \mathbb{P}^1$ is a G-cover, i.e. $\mathrm{Gal}(L/K) \cong G$, then there is a tower of covers

$$\begin{tikzcd} Y = Y_n \arrow[d] & \\ \vdots \arrow[d] & \vdots \\ Y_2 \arrow[d, "\mathbb{Z}/p\mathbb{Z}"'] & \\ Y_1 \arrow[d, "\mathbb{Z}/p\mathbb{Z}"'] & \\ Y_0 \arrow[d, "\mathbb{Z}/m\mathbb{Z}"'] & \\ \mathbb{P}^1 & \end{tikzcd}$$

wild part

tame part

where $G \cong \underline{\mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}}$ for $p \nmid m$.

A little more about ramification jumps, Swan conductors etc.:

$$j := \min\{\operatorname{ord}_x(f) \mid y^p - y = f\}$$

$=$ break in the ramification filtration

$$G \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots$$

$G_{j+1} \neq G_j$

$$G_i = \left\{\sigma \in G \;\middle|\; v\left(\frac{\sigma y}{y} - 1\right) \geq i\right\}$$

$y^p - y = f$

$B/A \cong A[y]/A$.

$$Sw := \frac{pj}{\ }$$

e ← ram. index of L/K

Both generalize to any G-cover $Y \to \mathbb{P}^1$ (and Sw generalizes much, much further — cf. the work of Kato, Saito, etc.)

$\boxed{\text{General Wild Covers}}$

Recall the tower of curves

$$
\begin{array}{ccc}
Y = Y_n & & \\
\downarrow & & \\
\vdots & & \vdots \\
| & &
\end{array}
$$

wild part

$$\begin{tikzcd} \vdots \arrow[d] \\ Y_2 \arrow[d, "\mathbb{Z}/p\mathbb{Z}"] \\ Y_1 \arrow[d, "\mathbb{Z}/p\mathbb{Z}"] \\ Y_0 \arrow[d, "\mathbb{Z}/m\mathbb{Z}"] \\ \mathbb{P}^1 \end{tikzcd}$$

tame part

where $G \cong \mathbb{Z}/p^n\mathbb{Z} \rtimes \mathbb{Z}/m\mathbb{Z}$ for $p \nmid m$.

One can generalize j to a sequence of ramification jumps

$$j_1, j_2, \ldots, j_n$$

where j_ℓ is the ℓth break

in the filtration

$$G \supseteq G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots$$

$$G_i = \{\sigma \in G \mid v(\tfrac{\sigma y}{y} - 1) \geq i\}$$

$y =$ unif. of $\hat{L}/\hat{K}$
@ ∞.

This definition is clumsy to work with (e.g. doesn't play nicely with subs and quotients simultaneously).

(Assume $m=1$, i.e. $G = \mathbb{Z}/p^n\mathbb{Z}$.)

More elegant description: by

Artin-Schreier-Witt theory, our

extension $L/K := k(y)/k(t)$

$\wp = F - 1$

is of the form

$\underline{y}^p - \underline{y} = \underline{f}$

$$L = K(\wp^{-1}(f))$$

Witt-vector valued

where f is a Witt vector-valued

rational function $f : \mathbb{P}^1 \dashrightarrow \mathbb{W}_n$.

As in the Artin-Schreier case,

étale $\mathbb{Z}/p^n\mathbb{Z}$-covers of $\mathbb{A}^1$ come from (wildly) ramified covers

of $\mathbb{P}^1$ branched at ∞,

Solution (Garuti 1999): there exists

a projective compactification

$$\overline{\mathbb{W}}_n \supseteq \mathbb{W}_n$$

such that rational maps

$$f : X \dashrightarrow \mathbb{W}_n$$

correspond to regular maps

$$X \longrightarrow \overline{\mathbb{W}}_n .$$

Garuti shows that the Swan conductor of $Y \to \mathbb{P}^1$ can be computed geometrically (when k is perfect):

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\mathbb{Z}/p^n"'] \arrow[dr, phantom, "\lrcorner\ \text{ét. loc.}"] & \overline{\mathbb{W}}_n \arrow[d, "\Phi_n \approx F-1"] \\ \mathbb{P}^1 \arrow[r, "f"] & \overline{\mathbb{W}}_n \end{tikzcd}$$

$$\mathrm{Sw}_\infty(Y/\mathbb{P}^1) = \deg f^*\mathcal{O}(1)$$

Work in progress w/ V. Thatte:

[illegible] Garuti's work to the case

Extend Garuti's work to the case when k is imperfect, using a generalization of "best f" for Witt vectors.

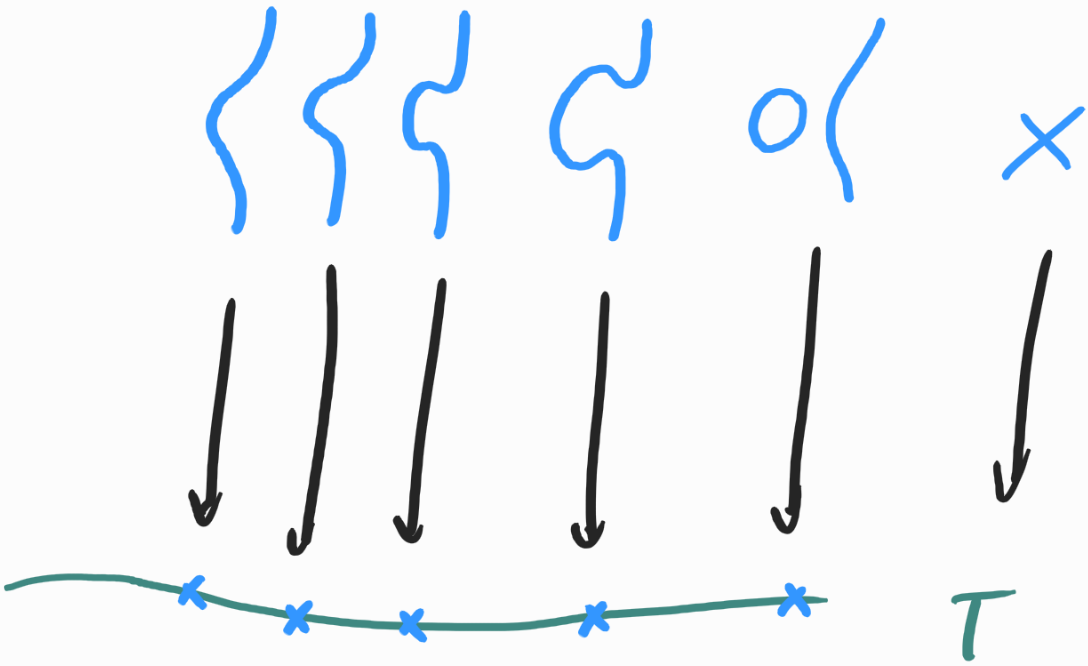
Moduli of Curves

Suppose we want to study these covers $Y \longrightarrow \mathbb{P}^1$ <u>in</u> <u>families</u>:

- a family of curves over a scheme T (or a T-curve) is a flat morphism

$$X \longrightarrow T$$

such that each fibre X_t is a (smooth projective) curve.



- a family of covers of curves (or a cover of T-curves) is a commutative triangle

$$Y \longrightarrow X$$

T

where all the fibres $Y_t \to X_t$ are covers of curves.

$\boxed{\text{Ex}}$

1. $\mathcal{M}_g$ $\qquad ASW_{cov_g}(\bar{k})$

2. P_d

3. ASW_g = ASW curves gen. g $\quad \mathbb{Z}/p^n \hookrightarrow \mathrm{Aut}(C)$

4. $ASWcov_g$ =

$$\begin{tikzcd} Y \arrow[r] \arrow[dr] & \mathbb{P}^1 \arrow[d] \\ & T \end{tikzcd}$$

moduli spaces

moduli stacks

Let $\mathrm{ASWcov}_g^{j} \subseteq \mathrm{ASWcov}_g$ denote the moduli stack of ASW covers of $\mathbb{P}^1$ with genus g and ramification jump sequence $j = (j_1, \dots, j_n)$.

That is, $\mathrm{ASWcov}_g^{j}(T)$ is the groupoid of T-covers $Y \xrightarrow{\varphi} \mathbb{P}^1$ with:

- $g(Y_t) = g$

- φ_t is branched at ∞ with ramification jumps $j_1, \dots, j_n$.

For $n=1$ (AS covers), the points of $AScov_g^j(T)$ are classified by maps to a weighted projective line:

$$\begin{tikzcd}
y^p - y = \frac{1}{t^j} \quad Y \arrow[r] \arrow[d] \arrow[rr, bend left] & \mathbb{P}(1,j) \arrow[r] \arrow[d, "\rho"] & \mathbb{P}^1 \arrow[d, "\rho"] \\
\mathrm{Proj}(k[t_0,t_1]) = \mathbb{P}^1 \arrow[r, "{[t_0 : t_1^j]}"] \arrow[rr, bend right, "f"'] & \mathbb{P}(1,j) \arrow[r, "\text{coarse}"] & \mathbb{P}^1
\end{tikzcd}$$

More generally, there is a stacky compactification

$$\overline{\mathrm{Hur}}_n(j_1,\dots,j_n) \longrightarrow \overline{\mathrm{Hur}}_n \supseteq \mathrm{Hur}_n$$

Work in progress: use $\overline{\mathrm{Hur}}_n(j_1,\dots,j_n)$ to describe points of ASWcov_g^j.

Thanks for your attention!

Questions?