Pretalk: Objective Linear Algebra

In the main talk, we'll see how to lift zeta functions

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

the Riemann zeta function

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s}$$

for a number field $K/\mathbb{Q}$

$$Z(X,t) = \exp\left[\sum_{n=1}^{\infty} \frac{\#X(\mathbb{F}_{q^n})}{n} t^n\right]$$

for a variety $X/\mathbb{F}_q$

$$= \sum_{\alpha \in Z_0^{\mathrm{eff}}(X)} 1\, t^{\deg(\alpha)}$$

to an "objective incidence algebra":

$$\begin{tikzcd} & \mathbb{N} \arrow[dl, "id"'] \arrow[dr] & \\ \mathbb{N} & & * \end{tikzcd}$$

$$\Big\downarrow \text{lift}$$

$$\zeta_{\mathbb{Q}}(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \longleftrightarrow (1, 1, 1, \dots) \in I(\mathbb{Q})$$

$$\begin{tikzcd} & I_K^+ \arrow[dl, "id"'] \arrow[dr] & \\ I_K^+ & & * \end{tikzcd}$$

$$\Big\downarrow \text{lift}$$

$$\zeta_K(s) = \sum_{\mathfrak{a} \subset \mathcal{O}_K} \frac{1}{N(\mathfrak{a})^s} \longleftrightarrow (1, 1, 1, \dots) \in I(K)$$

$$\begin{tikzcd} & Z_0^{eff}(X) \arrow[dl, "id"'] \arrow[dr] & \\ Z_0^{eff}(X) & & * \end{tikzcd}$$

$$\Big\downarrow \text{lift}$$

$$Z(X,t) = \sum_{\alpha \in Z_0^{eff}(X)} 1\, t^{\deg(\alpha)} \longleftrightarrow (1, 1, 1, \ldots) \in I(X)$$

The key idea is that

sets categorify numbers.

$\boxed{\text{Ex}}$ For natural numbers a and b,

let A and B be sets with

$\#A = a$ and $\#B = b$. Then

$$\#(A \amalg B) = a + b$$

$$\#(A \times B) = ab$$

$$\#\operatorname{Hom}(A, B) = b^a$$

$$\operatorname{Hom}(A \amalg C, B) \cong \operatorname{Hom}(A, B) \times \operatorname{Hom}(C, B) \quad \longleftrightarrow \quad b^{a+c} = b^a b^c$$

etc.

Objective linear algebra (due to Lawvere,

Menni et al.): linear algebra with cardinalities

Menni, et al.]: linear algebra with sets.

If the only scalars we're interested in are $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, we can lift many concepts from linear algebra to the category **Set** and recover original versions by taking cardinality:

Numerical	Objective
scalar a	set A
$+, \times$	$\amalg, \times$
ground field k	category **Set**
basis B	set B
vector v in the basis B	set map $v: V \to B$

scalar multiplication av	product map $A \times V \to V \to B$
addition $v+w$	sum $V \amalg W \to B$
vector space kB	slice category $\mathrm{Set}_{/B}$

$\boxed{\text{Ex}}$ The vector $v = (1, 4, 3)$ can be encoded by the objective vector

$$\begin{tikzcd} V = \{a_1, \underbrace{b_1, b_2, b_3, b_4}, \underbrace{c_1, c_2, c_3}\} \arrow[d, "v"'] \\ B = \{a, b, c\} \end{tikzcd}$$

Next: an objective matrix is a span

$$\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr, "t"] & \\ B & & C \end{tikzcd}$$

Think: a row of column vectors

OR a column of row vectors!

A vector $v: V \longrightarrow B$ can be viewed as

$$\begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ * & & B \end{tikzcd} \qquad \text{OR} \qquad \begin{tikzcd} & V \arrow[dl] \arrow[dr] & \\ B & & * \end{tikzcd}$$

a column vector $\qquad\qquad$ a row vector

To multiply a matrix by a vector,

take the pullback:

$$\begin{tikzcd}
 & W \arrow[dl] \arrow[dr, "s^*(v)"] & & \\
V \arrow[d] \arrow[dr, "v"] & & M \arrow[dl, "s"'] \arrow[dr, "t"] & \\
{*} & B & & C
\end{tikzcd}$$

$w = Mv$

$\boxed{\text{Ex}}$ We can represent the matrix

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

with the span

$$M = \left\{ \begin{matrix} a_1 & b_1, b_2 \\ c_1, c_2, c_3 & d_1, d_2, d_3, d_4 \end{matrix} \right\}$$

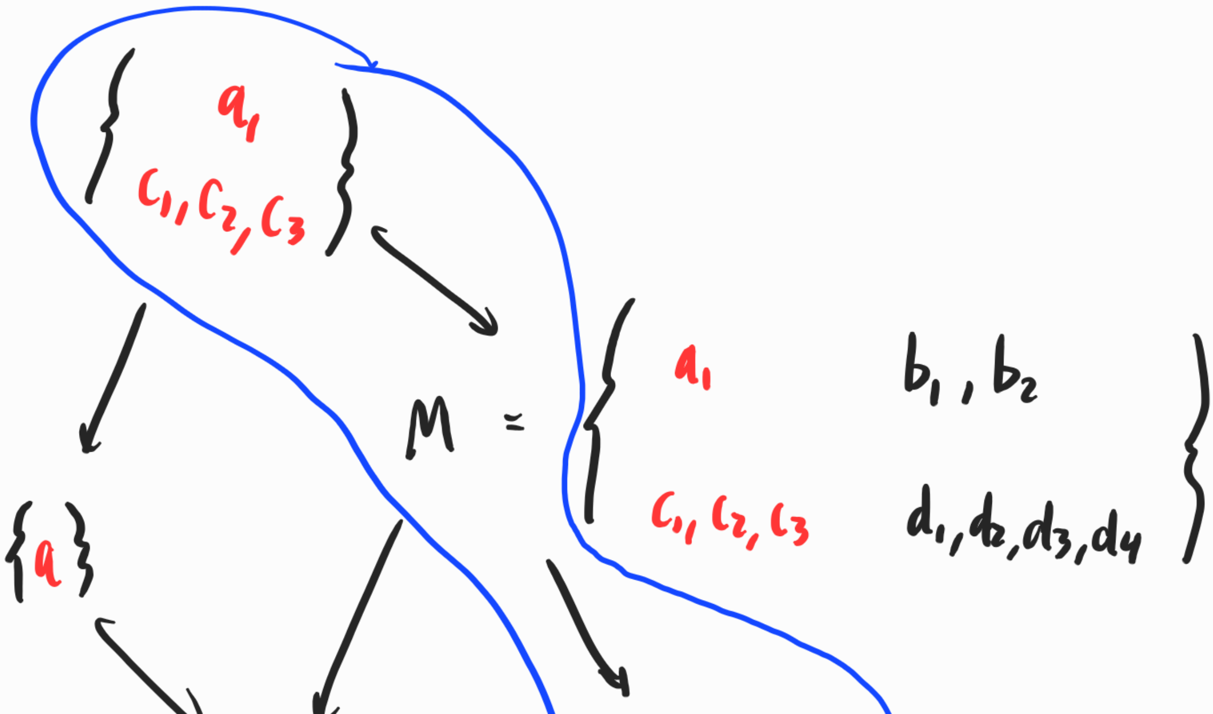
s

$B = \{a, b\}$ $\qquad\qquad$ $C = \{c, d\}$

To pick off the first column, multiply by the vector

$$\begin{tikzcd} V = \{a\} \arrow[d] & = \text{“}\begin{pmatrix}1\\0\end{pmatrix}\text{”} \\ B = \{a, b\} & \end{tikzcd}$$

using the diagram



$\{a, b\}$ $\qquad$ $\{c, d\} = $ "$\begin{pmatrix} 1 \\ 3 \end{pmatrix}$"

A matrix
$$\begin{tikzcd} & M \arrow[dl, "s"'] \arrow[dr, "t"] & \\ B & & C \end{tikzcd}$$
induces a linear functor $t_! s^* : Set_{/B} \longrightarrow Set_{/C}$ where

$$s^* : Set_{/B} \longrightarrow Set_{/M}$$

$$\left(\begin{tikzcd} V \arrow[d] \\ B \end{tikzcd} \right) \longmapsto \left(\begin{tikzcd} & s^*V \arrow[dl] \arrow[dr] & \\ V \arrow[dr] & & M \arrow[dl, "s"] \\ & B & \end{tikzcd} \right)$$

$$t_! : Set_{/M} \longrightarrow Set_{/C}$$

$$\left(\begin{tikzcd} X \arrow[d] \\ M \end{tikzcd} \right) \longmapsto \left(\begin{tikzcd} X \arrow[d] \arrow[dr, dashed] & \\ M \arrow[r] & C \end{tikzcd} \right)$$

This gives us a category LIN with

objects = slice categories $\mathrm{Set}/_B$

morphisms = linear functors $\mathrm{Set}/_B \to \mathrm{Set}/_C$.

More concepts that lift to OLA:

$$V \otimes W \rightsquigarrow \mathrm{Set}/_B \otimes \mathrm{Set}/_C := \mathrm{Set}/_{B \times C}$$

$$\mathrm{Hom}(V, W) \rightsquigarrow \mathrm{LIN}(\mathrm{Set}/_B, \mathrm{Set}/_C)$$

$$V^* \rightsquigarrow \mathrm{LIN}(\mathrm{Set}/_B, \mathrm{Set})$$

Q: What if we need scalars ot

Q: What if we need scalars other than $\mathbb{N}_0$?

Main talk: L-functions have complex coefficients in general.

Solution: replace the category Set with a (symmetric monoidal) category of spaces $\mathcal{S}$, leading to homotopy linear algebra (Gálvez-Carrillo, Kock, Tonks).

Key examples:

$\mathcal{S} = G_{\mathbb{Q}}\text{-Rep} \rightsquigarrow$ L-functions

$\mathcal{S} = \mathrm{Top} \rightsquigarrow$ Macdonald polynomials, etc.

$\mathcal{S} = \mathrm{Var}_k \rightsquigarrow$ motivic zeta functions

$\mathcal{S} = \mathrm{Stack}_k \rightsquigarrow$ motivic L-functions

+ more ?

$\mathcal{S} =$ locally compact abelian groups $\rightsquigarrow$ archimedean factors?