What is a Moduli Problem?

Loose, but useful definition: a moduli problem is a problem of the form "classify all objects of a certain type (maybe up to isomorphism, homotopy, etc.)" that has some geometric structure to it.

The geometric structure allows one to view the objects in the problem as points in a larger space, called a moduli space.

Ex:

	Moduli problem	Classified by	Moduli space	
	Circles in $\mathbb{R}^2$	radius + center	$\mathbb{R}_{>0} \times \mathbb{R}^2$	representable
	Circles up to isometry	radius	$\mathbb{R}_{>0}$	
	Lines through the origin in $\mathbb{R}^2$	unit vector up to ± 1	$\mathbb{P}^1_{\mathbb{R}}$	
more generally:	k-dim'l subspaces of $\mathbb{R}^n$	k-frame up to $GL_k(\mathbb{R})$	$Gr(k,n)$	
	Complex/hyperbolic structures on a surface of genus g	$6g-6$ Fenchel-Nielsen coordinates	$\mathbb{R}^{6g-6}$	
	Rank n vector bundles on X up to iso.	open cover $\{X_i\}$, $X_i \times \mathbb{R}^n \to X_i$, transition maps	$Gr(k, \infty)$	representable, but need $\sim$
	Principal G-bundles on X up to iso.	open cover $\{X_i\}$, $X_i \times G \to X_i$, transition maps	BG	
	Solutions to poly. equations	points on a variety	the variety	

More?

Today I want to talk about 3 famous moduli problems in AG:

(1) Lines on a smooth cubic surface

(2) Elliptic curves over $\mathbb{C}$

(3) Elliptic curves with torsion structure ($\rightsquigarrow$ modular forms!)

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(1) $\boxed{\text{A cubic surface}}$ is the set of solutions of a degree 3 equation (homogeneous) in 4 variables, e.g. the Clebsch surface

i.e. in $\mathbb{P}^3$

$$x^3 + y^3 + z^3 + w^3 = (x+y+z+w)^3.$$

Such a surface is smooth if it has no singularities, e.g. $x^3 = 0$ is singular.

<u>Theorem</u>: A smooth cubic surface contains exactly 27 lines.

(Cayley, 1849) That is, the moduli space for the moduli problem "classify lines on a smooth cubic surface" is a 0-dimensional variety with 27 components.

There is an elementary proof of this fact, but it isn't necessarily that enlightening.

Instead, I'd like to explain how to use the moduli problem approach to attack the question.

Key insight: lines in $\mathbb{P}^3$ are parametrized by a moduli space:

$$Gr(2,4) = \text{the Grassmannian of 2-dim'l subspaces in } \mathbb{C}^4$$

Strategy: view lines in $S \subset \mathbb{P}^3$ as <u>cohomology classes</u> in $H^\bullet(Gr(2,4);\mathbb{Z})$.

More precisely, there is a rank 2 vector bundle $V \to Gr(2,4)$ whose fibre over a line $\ell \subset \mathbb{P}^3$ is the space of homogeneous linear forms on ℓ.

The symmetric product $E = Sym^3(V)$ is a vector bundle over $Gr(2,4)$ whose fibre over ℓ is the homogeneous cubic polynomials on ℓ.

Let $s_F : Gr(2,4) \to E$ be the section that takes $\ell \mapsto F|_\ell$.

Then the zeroes of s_F are precisely the lines on $\{F=0\}$.

On the other hand, the zeroes of s_F are represented by the Chern class

$$c_4(E) \in H^8(Gr(2,4); \mathbb{Z})$$

and using Schubert calculus, one can compute

$$c_4(E) = 9c_2(V)\left(2c_1(V)^2 + c_2(V)\right) \quad \text{by splitting principle}$$

$$= 27\sigma \quad \text{for a Schubert cell } \sigma \text{ corresponding to a } \underline{\text{point}}.$$

Thus there are 27 "points" on the moduli space of lines ℓ on the surface $\{F=0\}$.

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(2) Complex elliptic curves

A 1-dimensional variety X over $\mathbb{C}$ is called a <u>Riemann surface</u> because it has the structure of a <u>2-dim'l manifold</u>; write $X(\mathbb{C})$ to denote this surface.

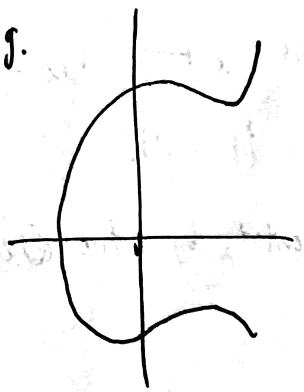
If X is smooth, projective and $X(\mathbb{C})$ is a sfc. of genus 1, with a specified basept. $0 \in X$, then $(X, 0)$ is called an <u>elliptic curve</u>.

Facts about elliptic curves:

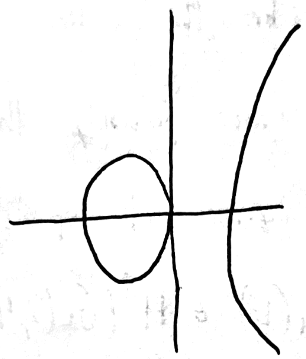
- they are given by a cubic equation called a <u>Weierstrass equation</u>:

$$X : y^2 = x^3 + Ax + B, \qquad A, B \in \mathbb{C}.$$

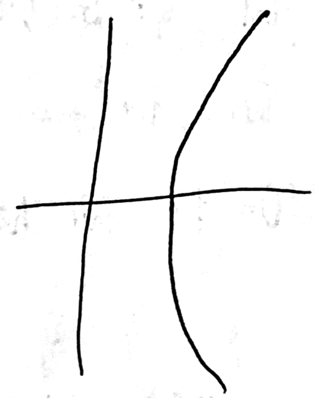
e.g.



$y^2 = x^3 - 3x + 3$

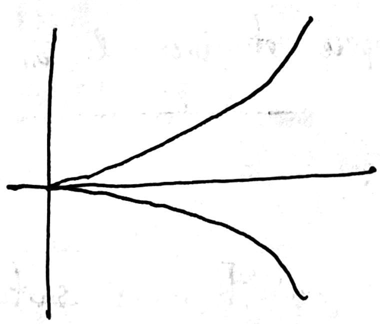


$y^2 = x^3 - x$

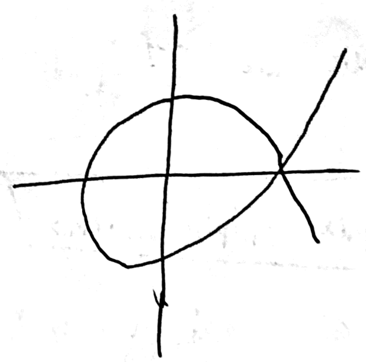


$y^2 = x^3 - 1$

non-smooth examples:



$y^2 = x^3$



$y^2 = x^3 - 3x + 2$

- X is smooth iff $\Delta := -16(4A^3 + 27B^2) = 0$ (called the <u>discriminant</u>)

How do we classify elliptic curves?

For an elliptic curve $X : y^2 = x^3 + Ax + B$, the <u>j-invariant</u> of X is

$$j = -1728 \frac{(4A)^3}{\Delta} = 1728 \frac{4A^3}{4A^3 + 27B^2}.$$

<u>Theorem</u>: Two elliptic curves X, X' are isomorphic iff $j = j'$.

<u>Corollary</u>: $\mathbb{C}$ is a moduli space for (iso. classes of) elliptic curves, written $\mathbb{A}^1_j$.

What does this geometric realization really do for us?

- elliptic curves live in a family: when two j's are close (in $\mathbb{A}^1_j$), the corresponding elliptic curves are close
- show visual at the end
- certain objects on $\mathbb{A}^1_j$ correspond to concrete things on ell. curves...

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(3) $\boxed{\text{Modular curves, modular forms, a zoo of other moduli problems}}$

There's another interpretation of $\mathbb{A}^1_j$ that is useful here:

$$\begin{tikzcd}
\mathbb{A}^1_j \arrow[rr, leftrightarrow] \arrow[d, leftrightarrow] & & \left\{ \begin{matrix} \text{complex lattices} \\ \text{up to iso.} \end{matrix} \right\} = \left\{ \begin{matrix} \Lambda \subseteq \mathbb{C} \\ \Lambda \cong \mathbb{Z}^2 \end{matrix} \right\} / \sim \arrow[dl, dashed] \\
\left\{ \begin{matrix} \text{elliptic curves} \\ \text{up to iso.} \end{matrix} \right\} & \Lambda \arrow[d, mapsto] & \\
& \mathbb{C}/\Lambda \cong \mathbb{R}^2/\mathbb{Z}^2 = \text{a torus!} &
\end{tikzcd}$$

$$\tau \longmapsto \mathbb{Z} \oplus \mathbb{Z}\tau$$

Alternatively, a lattice $\Lambda \subseteq \mathbb{C}$ may be written $\Lambda \cong \mathbb{Z} \oplus \mathbb{Z}\tau$ where τ (in $\mathfrak{h}$ = upper half-plane) is unique $/ PSL_2(\mathbb{Z}$

Set $Y = \mathfrak{h}/PSL_2(\mathbb{Z})$. Then Y is a moduli space for complex lattices up to iso., $Y \cong \mathbb{A}^1_j$ and so Y is also a moduli space for elliptic curves up to iso.

A differential form on Y is really a $PSL_2(\mathbb{Z})$-invariant differential form on $\mathfrak{h}$ — these are called <u>modular forms</u>.

Ex: The j-invariant itself may be viewed as a modular form.

The geometric description of this moduli problem using $Y = \mathfrak{h}/PSL_2(\mathbb{Z})$ suggests using other complex curves to solve related moduli problems.

<u>Fast facts</u>: (a) Every elliptic curve is an <u>abelian group</u>:

$$X(\mathbb{C}) = \mathbb{C}/\Lambda \leftarrow \text{quotient of ab. gps.}$$

(b) A <u>torsion point</u> on X is a point of order N for some $N < \infty$.

Explicitly, $P = \dfrac{\left(\frac{1}{N}\mathbb{Z} \oplus \mathbb{Z}\tau\right)}{\mathbb{Z} \oplus \mathbb{Z}\tau}$ has order N in $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$.

A <u>torsion subgroup</u> of X is a subgroup of finite order in $X(\mathbb{C})$.

(c) There are subgroups $\Gamma_0(N), \Gamma_1(N) \leq SL_2(\mathbb{Z})$ such that

$$Y_0(N) := \mathfrak{h}/\Gamma_0(N) \quad \text{and} \quad Y_1(N) := \mathfrak{h}/\Gamma_1(N)$$

are Riemann surfaces that parametrize elliptic curves with torsion points of order N ($Y_1(N)$) and with cyclic subgps. of order N ($Y_0(N)$).

(d) $\Gamma(N)$-invariant differential forms on $\mathfrak{h}$ are called <u>modular forms of level N</u>.