Fundamental Groups in Algebra and Geometry

Our goal is to draw the following picture

$$\begin{tikzcd} Y \arrow[d] \arrow[d, phantom, shift left=4ex, "\Big| \; G"] \\ X \end{tikzcd}$$

as many times as we can.

To each person in the room, this might have different meaning attached.

Examples: (1) covering space, with $G =$ deck transformations.

(2) field extension, with $G =$ Galois group

(3) morphism of varieties / schemes / curves with $G =$ Galois group of corresponding field extension

(4) extension of <u>differential</u> fields, i.e. fields with a derivation, with $G =$ Lie group $\rightsquigarrow$ differential Galois theory.

Let's review the most familiar case from topology.

<u>Def</u>: Let X be a nice space. A <u>cover</u> of X is a map $p: Y \to X$ that is a local homeomorphism, i.e. each fibre $p^{-1}(U) = \coprod U_\alpha$ and $p|_{U_\alpha} : U_\alpha \to U$ is a homeomorphism.

In AT, we classify equivalences classes of these covers by studying the category Cov_X : $\{Y \xrightarrow{p} X\}$

$$\mathrm{Hom}_X(Y, Z) = \left\{ \begin{tikzcd}[column sep=small] Y \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & X & \end{tikzcd} \right\}.$$

One defines a <u>universal cover</u> $\tilde{X} \to X$ to be the solution to the universal mapping problem in $\mathrm{Hom}_X(-,-)$ and then the <u>fundamental group</u> can be defined:

$$\pi_1^{top}(X) = \mathrm{Aut}_X(\tilde{X}).$$

$$\left.\begin{array}{c}\tilde{X}\\ \downarrow\\ X\end{array}\right]\pi_1^{top}(X)$$

(Alternatively, you can define this using loops but loops don't really make sense in Examples (2) – (4).)

Unfortunately, $\tilde{X}$ does not exist in algebraic categories, so we adopt a different perspective of π_1^{top}.

<u>Def</u>: The <u>fibre functor</u> over $x \in X$ is

$$\mathrm{Fib}_x : \mathrm{Cov}_X \longrightarrow \mathrm{Set}$$
$$(Y \xrightarrow{p} X) \longmapsto p^{-1}(x).$$

By the universal property of $\tilde{X}$, Fib_x is a representable functor:

$$\mathrm{Fib}_x(-) \cong \mathrm{Hom}_X(\tilde{X}_{\tilde{x}}, -).$$

choose a basept. in $\tilde{X}$

<u>Monodromy</u> is the action $\mathrm{Aut}_X(\tilde{X}) = \pi_1^{top}(X) \curvearrowright \mathrm{Hom}_X(\tilde{X}, Y)$ for any $Y \to X$ given by $\sigma . f = f \circ \sigma$ for $\sigma \in \mathrm{Aut}_X(\tilde{X})$, $f : \tilde{X} \to Y$.

<u>Theorem</u>: The induced map $\pi_1^{top}(X) \to \mathrm{Aut}(\mathrm{Fib}_x)$ is an isomorphism.

<u>Theorem</u>: Let X be connected and "nice" enough. Then there are equivalences of categories $\mathrm{Cov}_X \xrightarrow{\sim} \{\text{left } \pi_1(X)\text{-sets}\}$

$$\{\text{finite covers}\} \xrightarrow{\sim} \{\text{finite, cts. left } \widehat{\pi_1(X)}\text{-sets}\}$$

connected $\longleftrightarrow$ transitive

Galois $\longleftrightarrow$ quotients of $\widehat{\pi_1}$ by open normals.

Let's switch gears for a moment.

Fix a field k, an algebraic closure $\bar{k}$ and a separable closure $k \subseteq k^{sep} \subseteq \bar{k}$.

Set $G_k = \mathrm{Gal}(k^{sep}/k)$.

$\boxed{\text{For any finite, separable extension of fields } L/k, \ \mathrm{Hom}_k(L, k^{sep}) \text{ is a finite, cts., transitive } G_k\text{-set.}}$

<u>Theorem</u>: There is an anti-equivalence of categories

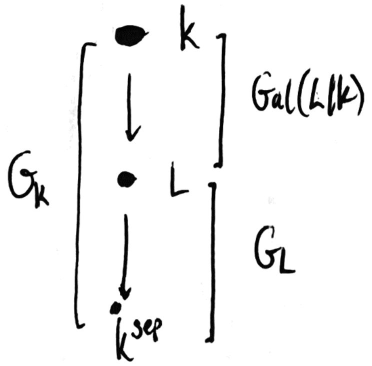
$$\{\text{finite, sep. } L/k\} \xrightarrow{\sim} \{\text{finite, cts., trans. } G_k\text{-sets}\}$$
$$L/k \longmapsto \mathrm{Hom}_k(L, k^{sep}).$$

What if we don't just want transitive sets?

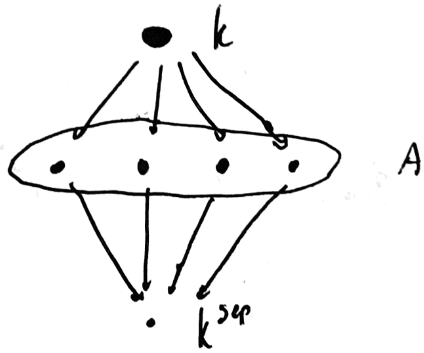
<u>Def</u>: A k-algebra A is a <u>finite étale k-algebra</u> if $A \cong L_1 \times \dots \times L_r$ for finite, separable extensions L_i/k.

<u>Corollary</u>: $\{\text{finite étale } k\text{-algebras}\} \xrightarrow{\sim} \{\text{finite, cts } G_k\text{-sets}\}$.

Topological interpretation:



or



Okay now I get to talk about some algebraic geometry.

Let X be a proper normal curve over a field k:

$(X, \mathcal{O}_X)$ is a ringed space

X - top. space $\boxed{\text{of dimension } 1}$

$\mathcal{O}_X$ - sheaf of rings

$X = \bigcup X_i$ where $\underbrace{X_i \cong \operatorname{Spec} A_i}_{\text{(integral affine curve)}}$ for A_i a f.g. k-algebra,

$\dim A_i = \operatorname{trdeg}_k \operatorname{Frac}(A_i) = 1$

A morphism of curves $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$ is a pair of maps

$$f : Y \to X$$
$$f^\# : \mathcal{O}_X \to f_* \mathcal{O}_Y \quad \text{(pushforward)}.$$

We need the following dictionary:

$$\{\text{integral affine curves}/k\} \xrightarrow{\sim} \{\text{f.g. domains } A/k,\ \dim A = 1\}$$

$$(X, \mathcal{O}_X) \longmapsto \mathcal{O}_X(X)$$
$$\boxed{Y \to X \longmapsto \mathcal{O}(X) \to \mathcal{O}(Y)}$$
$$\operatorname{Spec} A \longleftarrow A$$
$$\boxed{\operatorname{Spec} B \to \operatorname{Spec} A \longleftarrow A \to B.}$$

Ex (to keep in mind): $k = \mathbb{C}$, $X = \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

$\mathcal{O}_X(\mathbb{P}^1 \setminus \infty) = \mathcal{O}_X(\mathbb{C}) = \mathbb{C}[t]$ - poly. ring

$\operatorname{Spec} \mathbb{C}[t] = \{ m_p \mid p \in \mathbb{C} \}$ where $m_p = (t - p)$

For a curve X, let $k(X)$ be its function field:

- $k(X) := k(U) = \operatorname{Frac} k[U]$ for some affine open $U \subseteq X$
- $\dim X = \operatorname{trdeg}_k k(X) = 1$

<u>Theorem</u>: X proper normal curve, then

$$\{\underline{\text{finite}} \text{ morphisms } Y \to X\} \xrightarrow{\sim} \{\text{f.g. field extensions } L/k(X),\ \operatorname{trdeg}_k L = 1\}$$

$$Y \longmapsto k(Y).$$

(1) $f^{-1}(U)$ is affine
(2) $f_* \mathcal{O}(U)$ is f.g. $/\mathcal{O}(U)$

For a finite, <u>separable</u> morphism $Y \xrightarrow{f} X$, f is <u>étale</u> at $P \in X$ if

$$\mathcal{O}(Y)/P\mathcal{O}(Y) \cong \prod_{i=1}^{r} L_i \quad \text{for } L_i/(\mathcal{O}(X)/P) \text{ finite separable.}$$

$\boxed{\text{(i.e. } \mathcal{O}(Y)/P\mathcal{O}(Y) \text{ is f. ét.} / k(P)\text{).}}$

Fact: $f: Y \to X$ finite sep., then f is étale over a nonempty open $U \subseteq X$ (which is a dense set!) so we call $f: Y \to X$ a <u>finite branched cover</u>, $\boxed{\text{and } \underline{\text{Galois}} \text{ if } k(Y)/k(X) \text{ is Galois.}}$

Now we have:

$$\begin{array}{ccc} Y & & k(Y) \\ \downarrow & & | \\ X & & k(X) \end{array} \quad \operatorname{Gal}(k(Y)/k(X))$$

surjective w/
finite fibres $f^{-1}(P)$

In our example, curves are $Y \to \mathbb{P}^1_{\mathbb{C}}$ with $k(Y)/\mathbb{C}(t)$ finite, separable.

Now let $K = k(X)$, fix K^{sep}/K and let $U \subseteq X$ be open.

Set $K_U = \bigcup_{L \supseteq K} L$ over $(K^{sep} \supseteq L \supseteq K) \leftrightarrow (Y \to X$ étale over $U)$.

The <u>étale fundamental group</u> of $U \subseteq X$ is

$$\pi_1^{et}(U) = Gal(K_U/K).$$

Then we have: $\left\{ \begin{matrix} \text{covers } Y \to X, \text{ finite} \\ \text{sep. étale over } U \end{matrix} \right\} \xrightarrow{\sim} \{\text{finite cts left } \pi_1(U)\text{-sets}\}$.

To get some good examples, we need this dictionary:

$$\left\{ \begin{matrix} \text{proper normal curves} \\ \text{w/ finite } Y \to X \end{matrix} \right\} \longleftrightarrow \{\text{finite ext. } L/k(X)\} \longleftrightarrow \left\{ \begin{matrix} \text{proper holomorphic} \\ \text{maps of Riemann} \\ \text{surfaces } Y \to X(\mathbb{C}) \end{matrix} \right\}$$

* Riemann existence theorem

<u>Theorem</u>: For $U \subseteq X$ defined over $k = \mathbb{C}$, (or any $k = \bar{k}$ of char. 0),

$$\pi_1^{et}(U) \cong \widehat{\pi_1^{top}(U(\mathbb{C}))} \text{ where } \pi_1^{top}(U(\mathbb{C})) = \langle x_1, y_1, \dots, x_g, y_g, \gamma_1, \dots, \gamma_n \mid [x_1, y_1] \cdots [x_g, y_g]\gamma_1 \cdots \gamma_n = 1 \rangle$$

if $U(\mathbb{C}) \cong$ complex g-torus w/ n holes.

Examples: ① $X = \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. One can show any finite étale cover $Y \to \mathbb{P}^1$ is an isomorphism, so $\pi_1^{\text{ét}}(\mathbb{P}^1) = 1$.

just like S^2 ! (Riemann sphere)

② $X = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{C} \setminus \{0\}$

By Theorem, $\pi_1^{\text{ét}}(X) \cong \hat{\mathbb{Z}}$:

covers $X_n \longrightarrow \mathbb{P}^1 \setminus \{0, \infty\}$

$$\updownarrow$$

$$\mathbb{C} \longrightarrow \mathbb{C} \setminus \{0\}$$
$$t \longmapsto t^n$$

(just like S^1 in topology!)

③ $\pi_1^{\text{ét}}(\mathbb{P}^1 \setminus \{0, \infty, 1\}) \cong$ free profinite on two generators.

Fact: any finite simple group can be generated by 2 elts.

So every such G is a quotient of $\pi_1^{\text{ét}}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$!

⓪ $X = \operatorname{Spec} k = \{\text{pt.}\}$, fix $k^{sep} \supseteq k$ (like choosing basept. because $\operatorname{Spec} k^{sep} \to \operatorname{Spec} k$.

Then $\left\{ \begin{array}{c} \text{finite étale covers} \\ Y \to \operatorname{Spec} k \end{array} \right\} \xrightarrow{\sim} \{ \operatorname{Spec} A \mid \text{finite étale alg. } A \}$

Here, $G_k \left[\begin{array}{c} k^{sep} \\ | \\ A \\ | \\ k \end{array} \right.$ and so $G_k = \pi_1^{\text{ét}}(\operatorname{Spec} k)$.

Questions: (1) What's the analogue of $\tilde{X} \to X$?

(2) What is $\operatorname{Spec} \mathbb{Z}$ and why is it an analogue of ~~the circle S^1?~~ knot theory in S^3?

<u>knot:</u>

$S^1 \hookrightarrow S^3 = \mathbb{R}^3 \cup \{\infty\}$

$\pi_1(S^1) = \mathbb{Z}$

i.e. $S^1 = K(\mathbb{Z}, 1)$

<u>prime:</u>

$p \in \mathbb{Z}$

$\operatorname{Spec} \mathbb{F}_p \hookrightarrow \operatorname{Spec} \mathbb{Z}$

$\pi_1^{et}(\mathbb{F}_p) = \operatorname{Gal}(\overline{\mathbb{F}_p}/\mathbb{F}_p) \cong \hat{\mathbb{Z}}$

i.e. $\operatorname{Spec}(\mathbb{F}_p) = K(\hat{\mathbb{Z}}, 1)$.

(3) 3-manifolds M $\longleftrightarrow$ rings of algebraic numbers $\mathcal{O}_K$

(4) How do we study <u>abelian</u> extensions of $k(X)$ (aka class field theory)?