Stacks: What Are They Good For?

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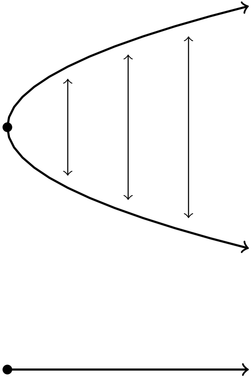
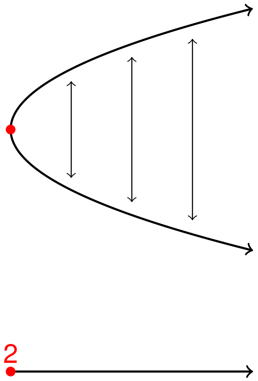
Introduction

Common problem: all sorts of information is lost when we consider quotient objects and/or singular objects.



Introduction

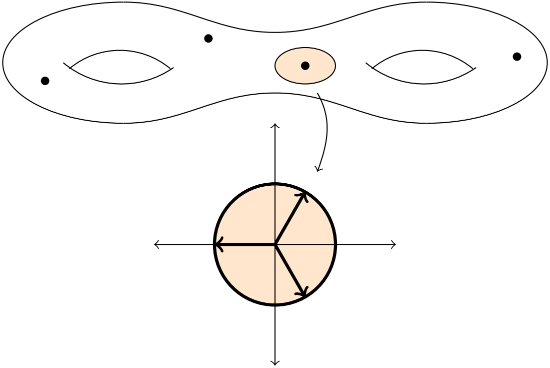
Solution: Keep track of lost information using *orbifolds* (topological and intuitive) or *stacks* (algebraic and fancy).



Complex Orbifolds

Definition

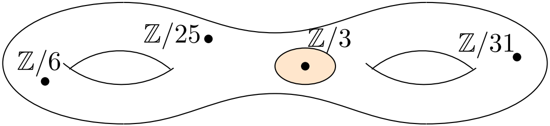
A **complex orbifold** is a topological space admitting an atlas $\{U_i\}$ where each $U_i \cong \mathbb{C}^n/G_i$ for a finite group G_i, satisfying compatibility conditions (think: manifold atlas but with extra info).



Algebraic Stacks

There's also a version of orbifold in algebraic geometry: an **algebraic stack**.

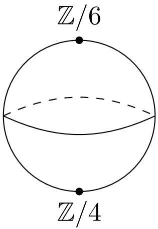
One important class of examples (Deligne–Mumford stacks) can be viewed as smooth varieties or schemes with a finite automorphism group attached at each point.



Running Example

Example

The (compactifed) moduli space of complex elliptic curves is a stacky $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ with a generic $\mathbb{Z}/2$ and a special $\mathbb{Z}/4$ and $\mathbb{Z}/6$.



We will delve further into this example in a moment.

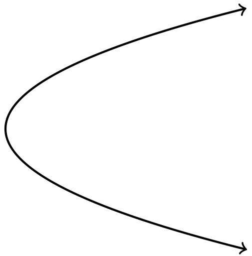
For now: “algebraic objects can be parametrized algebraically”, but we should keep track of automorphisms (e.g. to count properly!)

Moduli Problems

Loosely, a moduli problem is of the form "classify all objects of a certain type" admitting some natural geometric structure.

Example

The solutions to a polynomial equation $f(x_1, \ldots, x_n) = 0$ correspond to the points of an *algebraic variety*:



$y^2 - x = 0$

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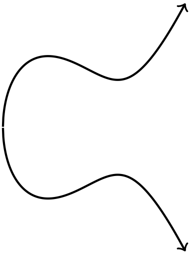

$$y^2 - x^3 + x - 1 = 0$$

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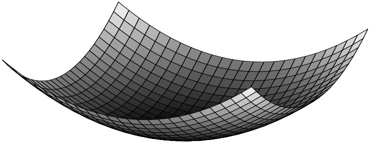

$$x^2 + y^2 - z - 1 = 0$$

Moduli Problems

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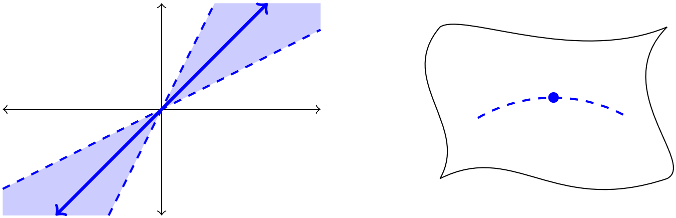
Example

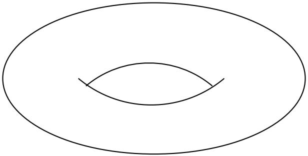
The moduli problem of finding r-dimensional subspaces of an n-dimensional vector space V is *represented* by a variety $\mathrm{Gr}(r, n)$ called the Grassmannian variety

e.g. $\mathrm{Gr}(1, n) = \mathbb{P}^n$, projective n-space.



Running Example: Elliptic Curves

An **elliptic curve** is a 1-dimensional complex manifold E (a *Riemann surface*) of genus 1 together with a specified basepoint O.



Fact: Every complex elliptic curve can be described by a *Weierstrass equation* $y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{C}$.

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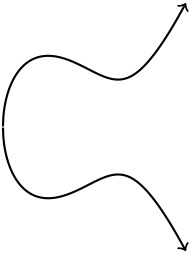

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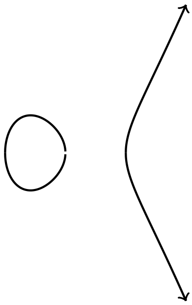

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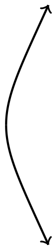

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To classify elliptic curves, we define an invariant

$$j(E) = 1728 \frac{4A^3}{4A^3 + 27B^2} \in \mathbb{C}.$$

Theorem

Two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$.

Corollary

The affine j-line $\mathbb{A}^1_j := \mathbb{C}$ is a moduli space for isomorphism classes of elliptic curves.

Running Example: Elliptic Curves

A **family of elliptic curves** over a complex manifold X is a holomorphic map $E \to X$ whose fibres are elliptic curves (and the distinguished points are picked out by a holomorphic section $O : X \to E$).

Let $\mathscr{M}_{1,1}(X)$ denote the set of isomorphism classes of (families of) elliptic curves over X. For any holomorphic map $Y \to X$, we get a map $\mathscr{M}_{1,1}(X) \to \mathscr{M}_{1,1}(Y)$ defined by pullback:

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

That is, $\mathscr{M}_{1,1}$ defines a functor $\texttt{CxMfld}^{op} \to \texttt{Set}$.

Running Example: Elliptic Curves

Question: Is $\mathscr{M}_{1,1} : \texttt{CxMfld}^{op} \to \texttt{Set}$ *representable*? That is, $\mathscr{M}_{1,1}(-) \cong \mathrm{Hom}(-, M)$ for a complex manifold M?

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Answer: Unfortunately, no. If there was, there would be a "universal elliptic curve" $E_0 \in \mathscr{M}_{1,1}(M)$ corresponding to $id_M \in \mathrm{Hom}(M, M)$ such that every elliptic curve $E \to X$ would be a pullback

$$\begin{array}{ccc} E & \xrightarrow{\tilde{f}} & E_0 \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & M \end{array}$$

for unique f and $\tilde{f}$.

However, no map $\tilde{f} : E \to E_0$ is unique: every elliptic curve $E : y^2 = x^3 + Ax + B$ has a nontrivial degree 2 automorphism $(x, y) \mapsto (x, -y)$. Further, if $j(E) = 0$ or 1728, $\mathrm{Aut}(E)$ is even larger.

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Partial Answer: The j-line $\mathbb{A}^1_j$ is a *coarse moduli space* for the moduli problem of elliptic curves. That is:

- there is a bijection $\mathscr{M}_{1,1}(*) \cong \mathrm{Hom}(*, \mathbb{A}^1_j) = \mathbb{C}$; and
- for any other complex manifold M and natural transformation $\mathscr{M}_{1,1}(-) \to \mathrm{Hom}(-, M)$, there is a unique map $\mathbb{A}^1_j \to M$ making the following diagram commute:

$$\begin{array}{ccc} \mathscr{M}_{1,1} & \longrightarrow & \mathrm{Hom}(-, \mathbb{A}^1_j) \\ & \searrow & \swarrow \\ & \mathrm{Hom}(-, M) & \end{array}$$

Example from Topology

Here's a related example from topology.

Example

Let G be a group and consider the principal G-bundle functor

$$\texttt{Top} \longrightarrow \texttt{Set}, \quad X \longmapsto \mathrm{Bun}_G(X) = \{\text{principal } G\text{-bundles}\}/\text{iso}.$$

There is a space BG which classifies G-bundles up to isomorphism:

$$\begin{array}{ccc} P & \xrightarrow{\tilde{f}} & EG \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & BG \end{array}$$

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Here, $(f, \tilde{f})$ are unique up to homotopy, i.e. there is a natural isomorphism $\mathrm{Bun}_G(-) \cong [-, BG]$. Want: unique on the nose.

Example from Topology

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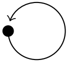
Example

To fix this, treat BG as a *groupoid*



$\mathrm{Bun}_G(X)$ can also be a viewed as a groupoid (remember the isomorphisms $P \xrightarrow{\sim} P'$ between G-bundles over X).

Want a natural isomorphism

$$\mathrm{Bun}_G(-) \cong \mathrm{Hom}_{\mathtt{Gpd}}(-, BG)$$

A Journey from Schemes to Stacks

The spaces I want to consider can be "described by algebra".

Motivation: Hilbert's Nullstellensatz says that the points of $\mathbb{C}^n$ are in bijection with certain ideals in a polynomial ring:

$$\mathbb{A}^n_{\mathbb{C}} := \mathbb{C}^n \longleftrightarrow \operatorname{MaxSpec} \mathbb{C}[z_1, \ldots, z_n]$$
$$P = (\alpha_1, \ldots, \alpha_n) \longmapsto \mathfrak{m}_P = (z_1 - \alpha_1, \ldots, z_n - \alpha_n).$$

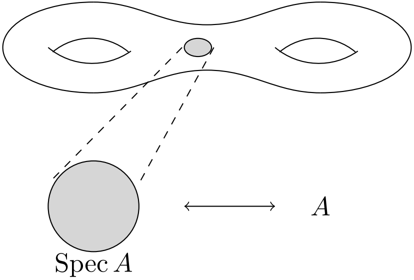
This is the jumping off point for algebraic geometry: for a general commutative ring A, there is a space $\operatorname{Spec} A$ defined by replacing "maximal ideals" with "prime ideals":

- Points of $\operatorname{Spec} A$ = prime ideals $\mathfrak{p} \subset A$
- Closed subsets of $\operatorname{Spec} A$ = "vanishing sets" $V(I) = \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\}$
- Sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} A$ with $\mathcal{O}(\operatorname{Spec} A) \cong A$ and stalks $\leftrightarrow$ localizations.

A Journey from Schemes to Stacks

Just as manifolds can be glued together from locally trivial patches, schemes are what we get by gluing together spaces built out of rings.

- An *affine scheme* is the space $\operatorname{Spec} A$ associated to a commutative ring A, together with its structure sheaf $\mathcal{O}$.
- A *scheme* is a locally ringed space which is locally affine, or can be obtained by "gluing" affine schemes (and their structure sheaves) in a compatible way.



Interlude: "Topology" is Flexible

Important: the gluing must play nicely with the chosen *topology* on $\mathrm{Spec}\, A$.

e.g. in ordinary topology, a cover $\{U_i \to X\}$ is a collection of open subsets $U_i \subseteq X$ with $\bigcup U_i = X$.

e.g. in the étale topology, a cover $\{U_i \to X\}$ is a collection of *étale morphisms* $f_i : U_i \to X$ with $\bigcup f_i(U_i) = X$.

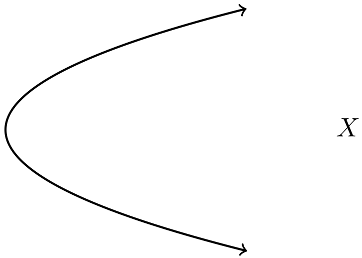
($f : U \to X$ is *étale* if it induces an isomorphism on tangent spaces. That is, étale is the algebraic analogue of local homeomorphism in topology.)

A Journey from Schemes to Stacks: The Functor of Points

To understand an object X (variety, scheme, etc.), it is enough to understand the set $X(T)$ of all maps $T \to X$ from all other objects T.

Example

Let X be a plane curve given by the equation $y^2 - x = 0$.



When $T = \operatorname{Spec} A$, $X(T) = \{$solutions to $y^2 - x = 0$ over $A\}$.

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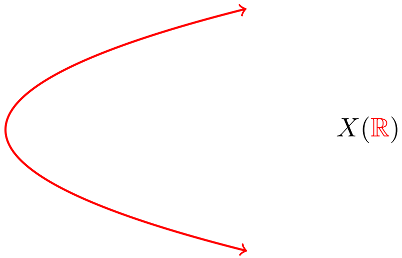

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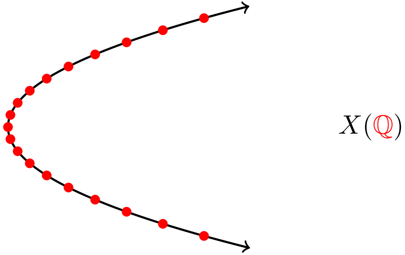

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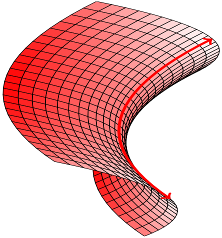


$X(\mathbb{C})$

When $T = \operatorname{Spec} A$, $X(T) = \{$solutions to $y^2 - x = 0$ over $A\}$.

A Journey from Schemes to Stacks: The Functor of Points

In other words, X determines a functor $X : \mathtt{AffSch}^{op} \to \mathtt{Set}$.

So every scheme X is like a moduli problem (super interesting in general: what do its points classify?)

Like we did with moduli problems, what happens if we instead have a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ with values in the category of *groupoids*.

Definition

A **stack** is a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ satisfying *descent*: for any étale cover* $\{U_i \to T\}$, the objects/morphisms of $\mathcal{X}(T)$ correspond to compatible objects/morphisms of $\{\mathcal{X}(U_i)\}$.

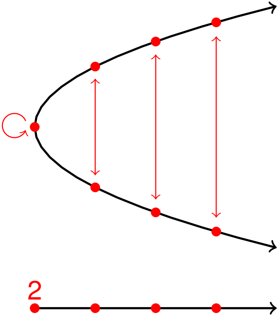
A Journey from Schemes to Stacks: The Functor of Points

Definition

A **stack** is a functor $\mathcal{X} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ satisfying *descent*.

Example

For our plane curve $X : y^2 - x = 0$, groupoids remember automorphisms like $(x, y) \leftrightarrow (x, -y)$



Quotients

Example

Let Y be a scheme and G be a group acting on Y. This determines a *quotient stack* $[Y/G] : \texttt{AffSch}^{op} \to \texttt{Gpd}$ defined by

$$[Y/G](T) = \left\{ \begin{array}{ccc} P & \xrightarrow{f} & Y \\ {\scriptstyle p}\downarrow & & \\ T & & \end{array} \right\}$$

where $p : P \to T$ is a principal G-bundle and $f : P \to Y$ is G-equivariant.

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Exercise: Why is this the right notion of quotient?

Quotients

Example

The *classifying stack* of a group G is the stack $BG : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ defined by

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Alternatively, $BG = [\bullet/G]$ and the universal G-bundle from topology is replaced by:

$$\begin{array}{ccc} P & \longrightarrow & \bullet \\ \downarrow & & \downarrow \\ T & \longrightarrow & BG \end{array}$$

Representability

Important Fact: For every scheme X, its functor of points $X(-) : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ is a stack. (Can you see why?) Write $X(T) = \mathrm{Hom}(T, X)$ for emphasis.

Definition

A stack $\mathcal{X}$ is **representable** if there is a natural isomorphism $\mathcal{X} \cong \mathrm{Hom}(-, X)$ for a scheme X.

So a moduli problem $\mathscr{M}$ is *representable* when $\mathscr{M} \cong \mathrm{Hom}(-, M)$ for a scheme M, called a *fine moduli space*.

Otherwise, we can still ask for $\mathscr{M}$ to be (represented by) a stack.

Example

For a group G, $\mathrm{Bun}_G(-) : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ is represented by a stack: $\mathrm{Bun}_G(-) \cong \mathrm{Hom}(-, BG)$.

Elliptic Curves Revisited

Let $\mathscr{M}_{1,1} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ be the moduli problem of elliptic curves,

$\mathscr{M}_{1,1}(T) =$ groupoid of maps of schemes $C \to T$, fibres $=$ ell. curves.

We saw that $\mathscr{M}_{1,1}$ is not represented by a scheme (well, a cx mfld but the same proof works). However:

Theorem

$\mathscr{M}_{1,1} : \mathtt{AffSch}^{op} \to \mathtt{Gpd}$ *is a stack.*

Next: let's explore the geometry of the stack $\mathscr{M}_{1,1}$.

The Point(s) of Stacks

In set theory, a point is rigid: it's just a one-element set $\{x\}$.

In topology, a point could be: a set-theoretic point OR a contractible space (a pt. up to homotopy).

In algebraic geometry, points come in many different flavors:

- For any field k, the underlying space of $\operatorname{Spec} k$ is a point. Field extensions $L/k \longleftrightarrow$ nontrivial maps of points $\operatorname{Spec} L \to \operatorname{Spec} k$.
- (Nilpotents are invisible) The underlying space of $\operatorname{Spec} k[x]/(x^2)$ is also a point, but $k[x]/(x^2) \not\cong k$ so the respective schemes are distinct.
- (Points can be dense) If A is an integral domain with fraction field K, the image of the corresponding map $\operatorname{Spec} K \hookrightarrow \operatorname{Spec} A$ is called the *generic point* of $\operatorname{Spec} A$. It is dense in the topology on $\operatorname{Spec} A$.

The Point(s) of Stacks

Definition

A **point** of a scheme X is an equivalence class of morphisms $x : \operatorname{Spec} k \to X$, where k is a field, and where two points $x : \operatorname{Spec} k \to X$ and $x' : \operatorname{Spec} k' \to X$ are equivalent if there exists a field $L \supseteq k, k'$ making the following commute:

$$\begin{array}{ccccc} & & \operatorname{Spec} k & & \\ & \nearrow & & \searrow^{x} & \\ \operatorname{Spec} L & & & & \mathcal{X} \\ & \searrow & & \nearrow_{x'} & \\ & & \operatorname{Spec} k' & & \end{array}$$

The Point(s) of Stacks

Replacing the scheme X with a stack $\mathcal{X}$ yields the same definition.

Definition

The **automorphism group** of a point $x : \operatorname{Spec} k \to \mathcal{X}$ of a stack $\mathcal{X}$ is the pullback G_x in the diagram

$$\begin{array}{ccc} G_x & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow (x,x) \\ \mathcal{X} & \xrightarrow{\Delta_{\mathcal{X}}} & \mathcal{X} \times \mathcal{X} \end{array}$$

A **stacky point** of $\mathcal{X}$ is a point with $G_x \neq 1$.

The Moduli Stack $\mathscr{M}_{1,1}$

Proposition

Let E be an elliptic curve over $\mathbb{C}$. Then $\mathrm{Aut}(E)$ is a finite cyclic group, and more specifically,

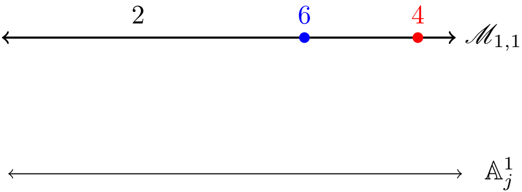
$$\mathrm{Aut}(E) = \begin{cases} \mathbb{Z}/6\mathbb{Z}, & \text{if } j(E) = 0 \\ \mathbb{Z}/4\mathbb{Z}, & \text{if } j(E) = 1728 = 12^3 \\ \mathbb{Z}/2\mathbb{Z}, & \text{if } j(E) \neq 0, 1728. \end{cases}$$

Theorem

Let $\mathscr{M}_{1,1}$ be the moduli stack of elliptic curves. Then $\mathscr{M}_{1,1}$ is a "stacky" $\mathbb{A}^1$ with automorphism group $\mathbb{Z}/2\mathbb{Z}$ at every point except two, which have automorphism groups $\mathbb{Z}/4\mathbb{Z}$ and $\mathbb{Z}/6\mathbb{Z}$. Further, there is a map $\mathscr{M}_{1,1} \to \mathbb{A}^1_j$ which is a bijection on geometric points and is universal with respect to maps $\mathscr{M}_{1,1} \to M$ where M is a scheme.

The Moduli Stack $\mathscr{M}_{1,1}$

That is, $\mathscr{M}_{1,1}$ looks like



The Moduli Stack $\mathscr{M}_{1,1}$

Utility of $\mathscr{M}_{1,1}$ over $\mathbb{A}^1_j$:

- Stores info about automorphisms that may even be invisible over subfields of $\mathbb{C}$, e.g. $\mathbb{Q}$.
- Classical objects (e.g. modular forms) have geometric interpretation over a nice compactification $\overline{\mathscr{M}_{1,1}}$ (e.g. as sections of line bundles), vs. ad hoc and awkward interpretation over $\mathbb{P}^1_j = \overline{\mathbb{A}^1_j}$.
- Counting formulas with fractional coefficients “come from geometry”.
- Accommodates extra structure like level structure (elliptic curves with torsion), polarizations, etc.
- “Lifts to topology” as a spectral (Deligne–Mumford) stack, leading to topological modular forms.

Higher Genus Curves

All of this generalizes to curves of genus $g \geq 2$:

- Let $\mathscr{M}_g$ be the moduli functor sending $T \mapsto$ the groupoid of relative curves $C \to T$ with genus g fibres.
- Then $\mathscr{M}_g$ is a stack that is not representable by a scheme.
- But it has a coarse moduli space M_g which has been well-studied.
- $\dim \mathscr{M}_g = \dim M_g = 3g - 3$ (compare to hyperbolic geometry).
- Marked version has deep ties to number theory, e.g. $\chi(\mathscr{M}_{g,1}) = \zeta(1 - 2g)$.

Thank you!

https://www.desmos.com/calculator/ialhd71we3