Moduli Problems + Root Constructions in Algebraic Geometry

Loose definition: a moduli problem is of the form "classify all objects of a certain type (maybe up to isometry, isomorphism, homotopy, etc.)" that has a certain geometric structure.

The geometric structure allows us to view objects to be classified as points in a larger space, called a moduli space.

Examples:

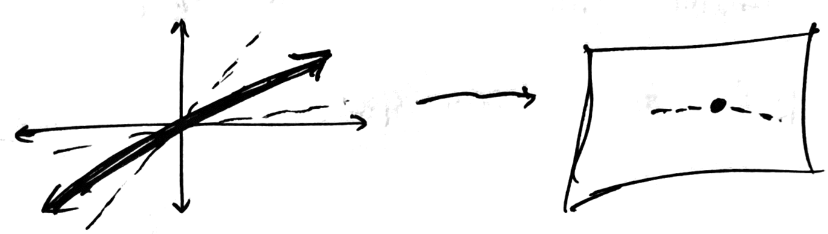
	Moduli problem	Classified by	Moduli space
	Circles in $\mathbb{R}^2$	radius + center	$\mathbb{R}_{>0} \times \mathbb{R}^2$
	Circles up to isometry	radius	$\mathbb{R}_{>0}$
	Lines through the origin in $\mathbb{R}^2$	unit vector up to ± 1	$\mathbb{RP}^1$ (or $\mathbb{P}^1_{\mathbb{R}}$)
more generally:	k-dim'l subspaces of $\mathbb{R}^n$	k-frame up to $GL_k(\mathbb{R})$	Grassmannian $Gr(k,n)$
	Complex/hyperbolic structures on a surface of genus g	$6g-6$ Fenchel-Nielsen coordinates	$\mathbb{R}^{6g-6}$
	Rank n vector bundles on X up to iso.	locally trivial $\begin{matrix} U_i \times \mathbb{R}^n \\ \downarrow \\ U_i \subseteq X \end{matrix}$ + data	infinite Grassmannian $Gr(n,\infty)$
	Principal G-bundles on X up to iso.	locally trival $\begin{matrix} U_i \times G \\ \downarrow \\ U_i \end{matrix}$ + data	BG
	Solutions to poly. eqn's	points on a variety	the variety

MORE?

$\boxed{\text{A cubic surface}}$ is described by a degree 3 homogeneous equation in 4 variables, e.g. the Clebsch surface

$$x^3 + y^3 + z^3 + w^3 = (x+y+z+w)^3 \text{ in } \mathbb{P}^3$$

projective n-space : the space of lines through the origin in $\mathbb{C}^{n+1}$



A cubic surface is smooth if it has no singularities, e.g. $x^3 = 0$ singular.

Theorem (Cayley–Salmon, 1849) : A smooth cubic surface contains exactly 27 lines.

(The moduli space classifying lines on a cubic sfc. is 27 discrete pts.)

Modern proof : $\{\text{lines in } \mathbb{P}^3\} \longleftrightarrow \{\text{points of the Grassmannian } Gr(2,4)\}$

2 dim'l subspaces of $\mathbb{C}^4$

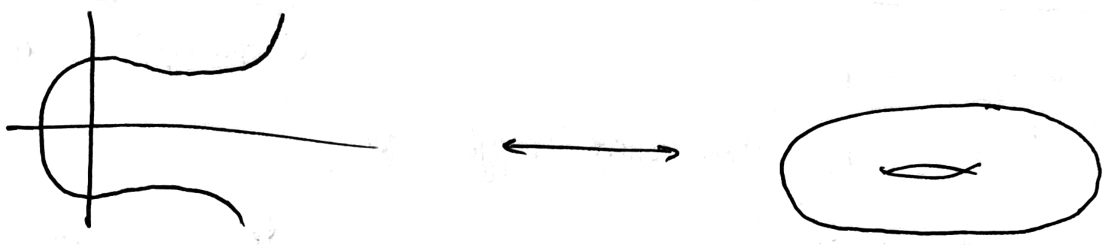
Then lines on a cubic $S \subseteq \mathbb{P}^3$ can be computed combinatorially using:

- Chern classes in $H^\bullet(Gr(2,4))$
- Schubert calculus = cellular decomp. of $Gr(2,4)$.

Elliptic curves

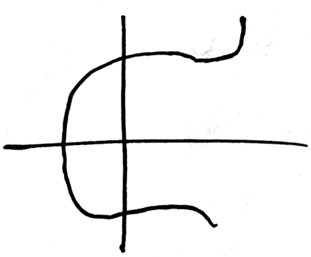
A 1-dimensional complex manifold is called a <u>Riemann surface</u> because it has the structure of a 2-dim'l ~~real~~ manifold.

If X is a (smooth, proper) Riemann surface of genus 1 (with a specified basept. $O \in X$,) then (X, O) is called an elliptic curve.

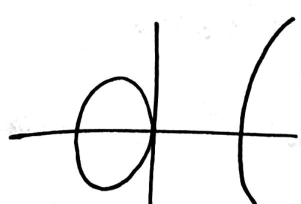


Fact: every elliptic curve can be described by a <u>Weierstrass equation</u>

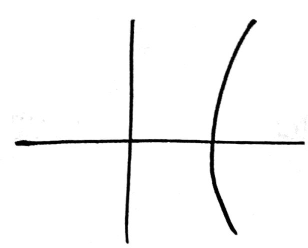
$$y^2 = x^3 + Ax + B, \quad A, B \in \mathbb{C}$$



$y^2 = x^3 - 3x + 3$

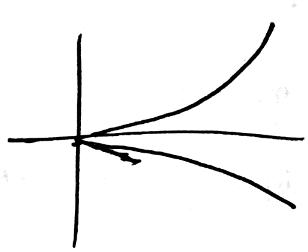


$y^2 = x^3 - x$

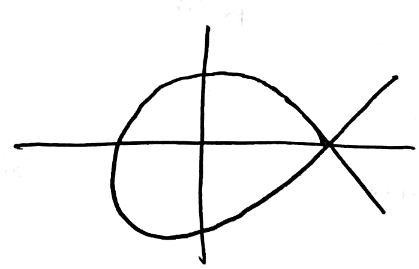


$y^2 = x^3 - 1$

some non-smooth equations:



$y^2 = x^3$



$y^2 = x^3 - 3x + 2$

The j-invariant of X is the complex number

$$j = 1728 \frac{4A^3}{4A^3+27B^2} \leftarrow (\neq 0 \text{ if } X \text{ smooth})$$

Theorem: Two elliptic curves are isomorphic iff they have the same j-invariant.

Cor: $\mathbb{C}$ is a moduli space for isomorphism classes of elliptic curves, written $\mathbb{A}^1_j$.

Consequences:
- elliptic curves live in a geometric family: when we wiggle j, the curve "wiggles" too
- certain objects on $\mathbb{A}^1_j$ correspond to concrete things on elliptic curves.

Modular Forms

A modular form is a function $f(z)$ on the upper half-plane

$$\mathfrak{h} = \{z = x+iy \in \mathbb{C} \mid y > 0\} \subseteq \mathbb{C}$$

that is "invariant" under the $SL_2(\mathbb{Z})$-action

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}$$

in the sense that $f(z) = (cz+d)^{-2k} f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right)$ for some $k \in \mathbb{Z}$,

+ some analytic conditions.

These are highly important mathematical objects:
- encode a wealth of number theoretic information
- connections to lattice theory, analysis, topology, string theory, etc.
- key ingredient in Wiles' proof of FLT
- just really cool!

Geometric interpretation: modular forms $\longleftrightarrow$ SL_2-invariant differential forms

$$f(z) \longleftrightarrow f(z)\,dz^k$$

If you haven't seen differential forms before, think of them as

$$\omega = (\text{function}) \cdot dx \wedge dy \wedge dz \wedge dw$$

or even more concretely as things that get integrated:

$$\int \omega = \int f(x,y)\,dx\,dy.$$

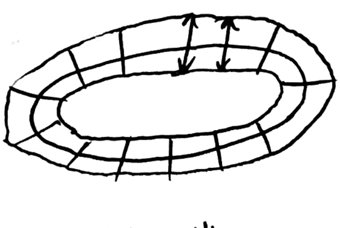
Abstractly, differential forms arise as <u>sections</u> of certain <u>line bundles</u>...

Let me explain these if you're unfamiliar.

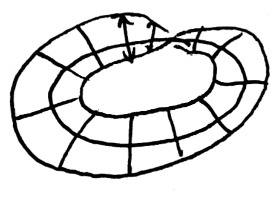
A <u>line bundle</u> on a space X is the continuous assignment of a 1-dim'l cx vector space at each pt. $x \in X$:

Useful for studying topology

Ex: Two line bundles on the circle



"trivial"



"twisted"

A <u>section</u> of a line bundle is a continuous of vector at each $x \in X$.

e.g. choose the core circle in each of the abaci

Fact: differential forms ($f dx \wedge dy$) are sections of a certain line bundle on X, $\Omega^k(X) = \Lambda^k T^* X$.

Then modular forms $\longleftrightarrow$ sections of the line bundle $\Omega^k(X)$ where $X = \mathfrak{h}/SL_2(\mathbb{Z})$. (Not quite...)

(Notice: $\mathfrak{h}/SL_2(\mathbb{Z}) \cong \mathbb{C} = \mathbb{A}^1_j$ so <u>modular</u> forms are examples of objects that can be computed on the <u>moduli</u> space of elliptic curves.)

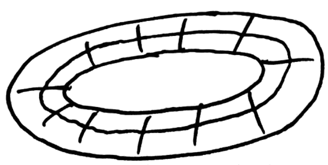
Root constructions

To compute spaces of modular forms (i.e. dimensions/bases of the vector space of modular forms), it is important to "unwind" the line bundles $\Omega^k(X)$ by taking roots.

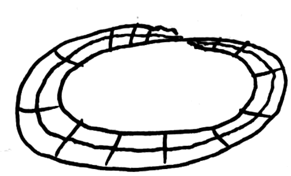
For example, evaluating $\omega = f\, dz$ by pulling back along the degree n map $\mathbb{C}^* \to \mathbb{C}$, $z \mapsto z^n$, gives information about the zeroes and poles of ω.

In general, the relevant question is: if (L, s) is a line bundle $L \to X$ with a section $s : X \to L$, can we find another pair (E, t) such that $L = E^{\otimes n}$ and $s = t^n$?

Ex: Line bundles on a circle



take an nth root at every pt.



if $n=2$, must choose $\pm\sqrt{\ }$ and this can't be done consistently!

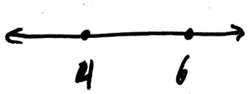
ASIDE:

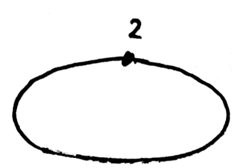
Other things you can take roots of in math:

- negative numbers $\rightsquigarrow$ complex numbers
- matrices, linear operators like derivatives
- topological spaces (what operations make sense? $\#$, $\wedge$, etc.)
- particles in physics

So the problem of rooting (L,s) can't be solved in general.

Solution: pass to a <u>stack</u>, which is just like the original space X but with extra information attached at certain points:





Fact: Over $\mathbb{C}$, every line bundle over X has an nth root on some stacky version of X — <u>where "line bundle" has to be subtly reinterpreted.</u>

Idea: at points where L twists, replace with a stacky point of that order by pulling back locally along $z \longmapsto z^n$.

Such a construction is called a <u>root stack</u>, and they help one compute dimension formulas for modular forms.

If k is a field of char. $p > 0$, $x \longmapsto x^p$ is problematic...

$$\boxed{\text{e.g. its derivative is } 0 \text{ but it's a nontrivial map}}$$

<u>Construction (K.)</u> : To take a pth root of a line bundle $L \longrightarrow X$ in char. p, take pullbacks along <u>Artin-Schreier maps</u>

$$x \longmapsto x^p - x.$$

The resulting object is called an <u>Artin-Schreier root stack</u>.

<u>Theorem (K.)</u> : Every cyclic degree p cover of curves $Y \longrightarrow X$ in char. p factors through an AS root stack.