What is a Moduli Problem?

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Begonia

Poppy

Today's lesson plan :

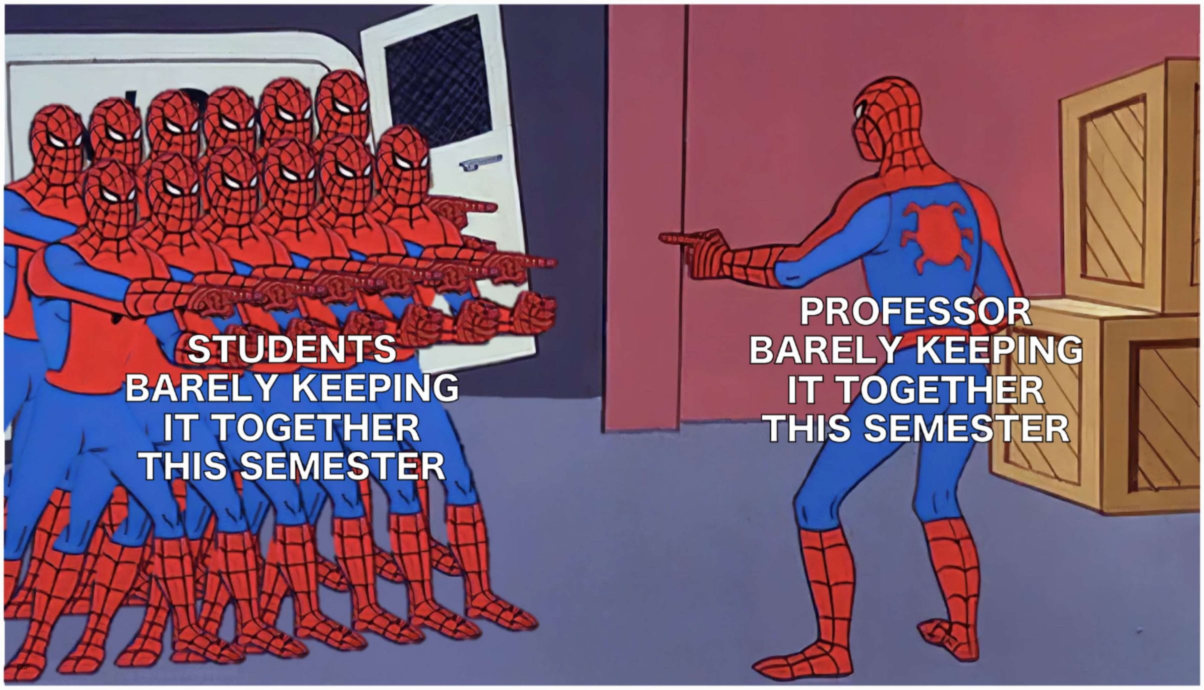
By the way, I just finished grad school last year.

Ha ha as if I ever looked that good! It was more like:

me c. 2015

me 2020 (-present)

Okay one more (for now):



Introduction

loose definition: a **moduli problem** is a problem of the form

"classify all objects of a certain type".

classify all objects ...

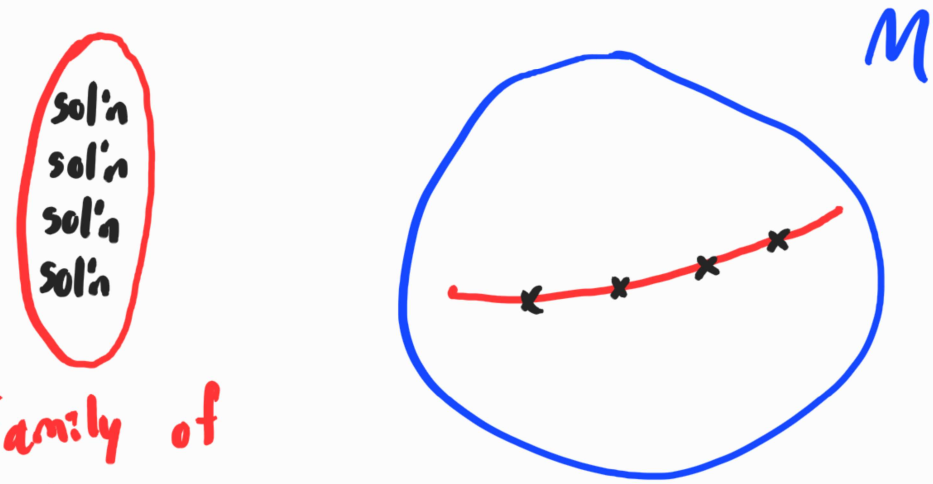
usually + "up to equivalence"

In many situations, the solutions to such problems can be parametrized geometrically — we can view the solutions themselves as points in some parameter space, called a **moduli space**.

The utility of this is:

- if the space M has any extra structure (e.g. a metric, a group structure, the structure of a variety), then this sheds light on all solutions simultaneously.
- most powerfully, we can produce families of solutions by picking meaningful subsets of the moduli space

of the moduli space.



family of solutions parametrized by a curve

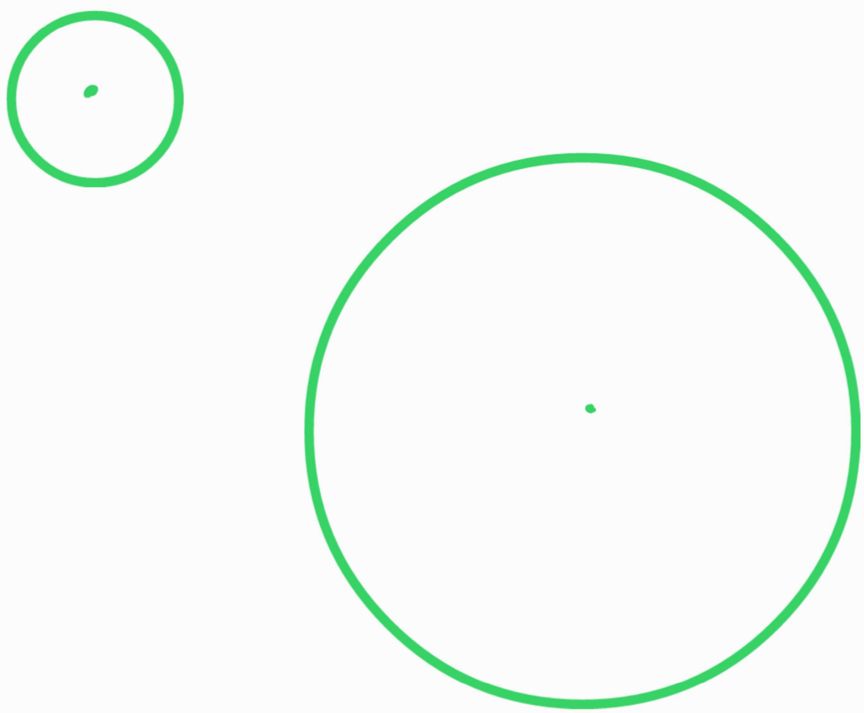
Elementary Examples

Here are some familiar examples:

1. Circles in $\mathbb{R}^2$ $\quad (x-h)^2+(y-k)^2=r^2$

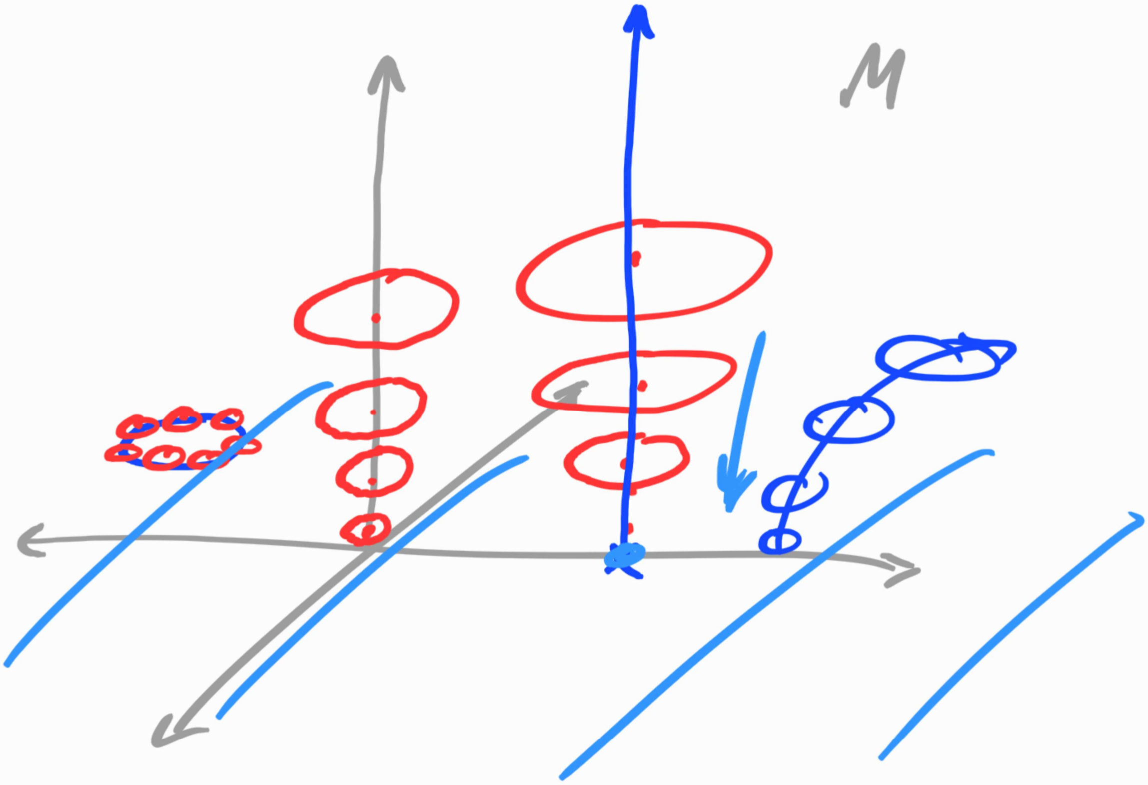
   $\underline{(h,k)}$, $\underline{r}$

Every circle $C \subseteq \mathbb{R}^2$ is uniquely specified by center + radius.



A moduli space for circles in $\mathbb{R}^2$

is $\underbrace{\mathbb{R}^2}_{\text{center}} \times \underbrace{\mathbb{R}_{>0}}_{\text{radius}} = \mathbb{R}^2 \times (0,\infty)$

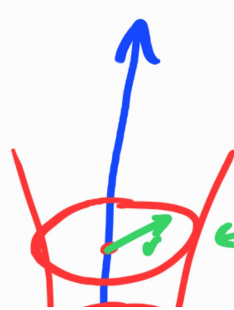


② Circles in $\mathbb{R}^2$ up to isometry

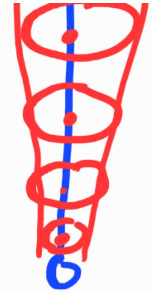
Two circles $C_1, C_2 \subseteq \mathbb{R}^2$ are isometric if they have the same <u>size</u>.

So circles up to isometry are classified by <u>radius</u>.

A moduli space for circles up to isometry is



the family of



all radius
r circles

3 Lines in $\mathbb{R}^2$

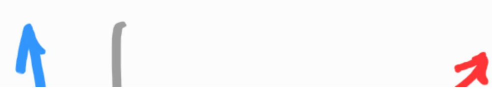
A line in $\mathbb{R}^2$ is given by an equation $ax + by = c$, so we can think of

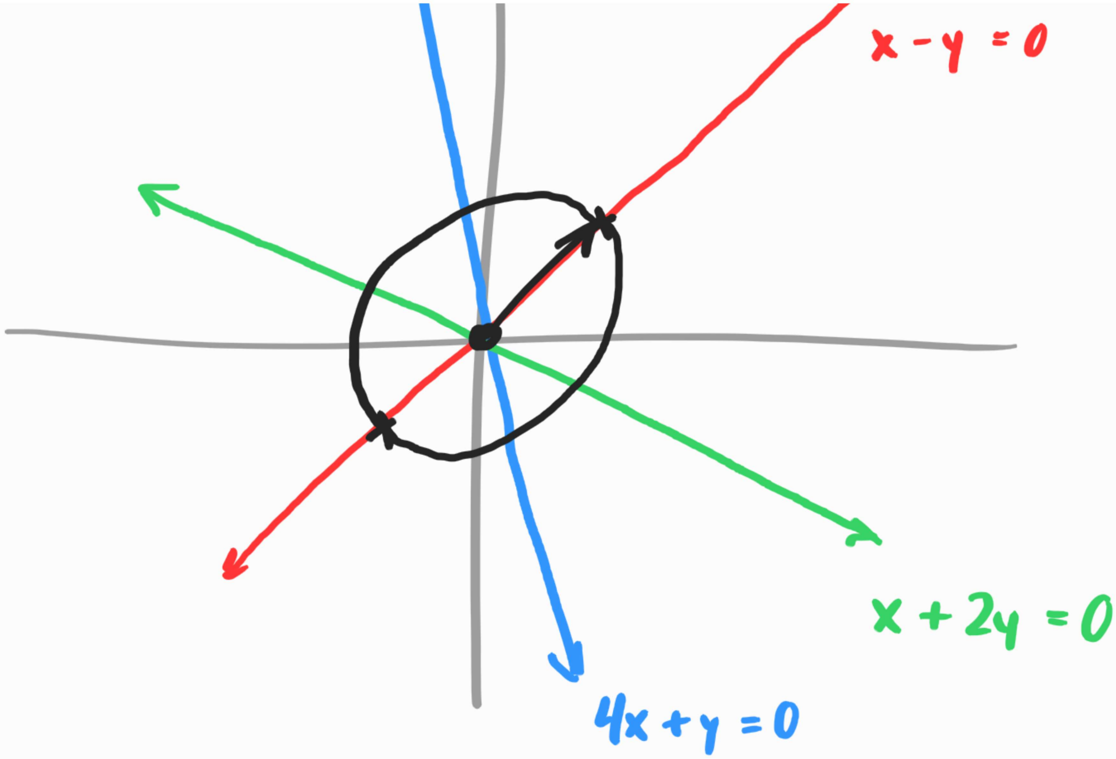
$$\mathbb{R}^3 = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}.$$

BUT this misses when two equations represent the same line

e.g. $(x + y = 1) \simeq (2x + 2y = 2)$

To make things simpler, let's just consider lines through $(0,0)$:





$L: \quad ax + by = 0$

A line through the origin is <u>uniquely</u> specified by

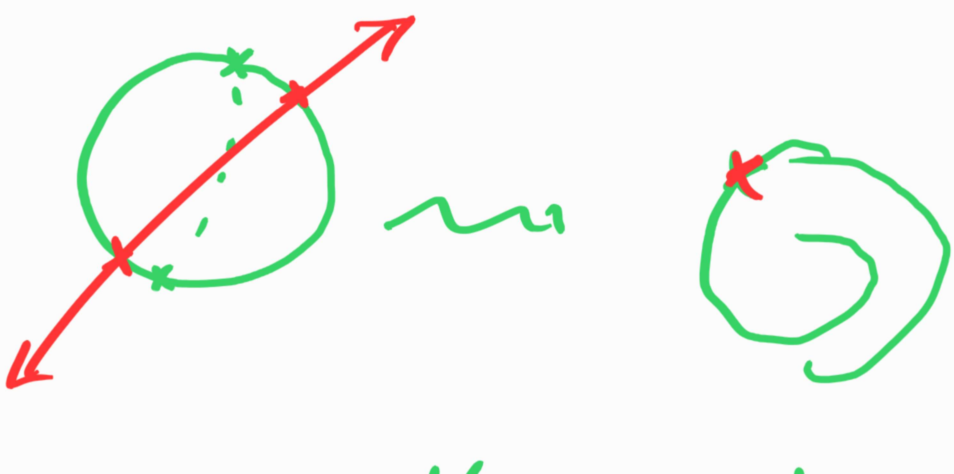
$ax + by = 0 \rightsquigarrow$ slope $-\frac{a}{b}$

pick (x, y) on S^1 up to

± 1

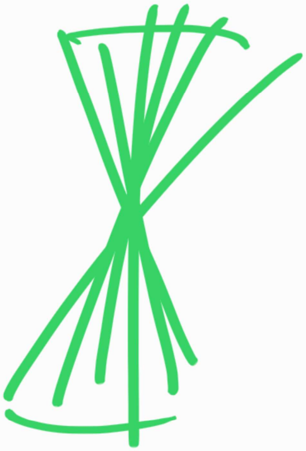
A moduli space of lines through the origin is

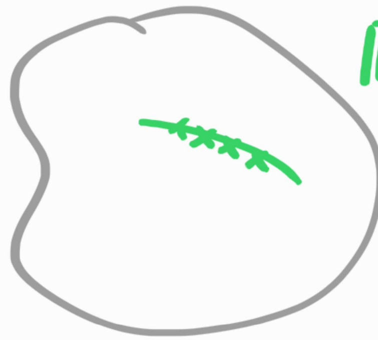
$S^1/\sim$ = pts. on circle up to being antipodes



$S^1/\sim \; = \; \mathbb{P}^1$

projective line





$\mathbb{P}^1$

4 More generally,

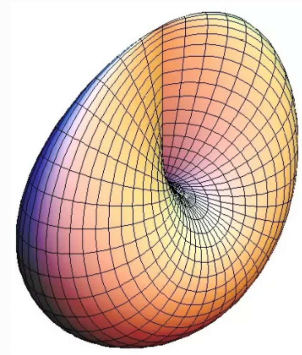
$$\left\{\begin{matrix}\text{lines through} \\ \vec{0} \text{ in } \mathbb{R}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{unit vector} \\ \text{up to } \pm 1\end{matrix}\right\}$$

$$\updownarrow$$

$$\left\{\begin{matrix}\text{points in} \\ \mathbb{P}^{n-1}\end{matrix}\right\}$$

where $\mathbb{P}^{n-1} = S^{n-1} / \substack{\text{identify} \\ \text{antipodes}}$

$\rightsquigarrow$



$\mathbb{P}^{n-1}$

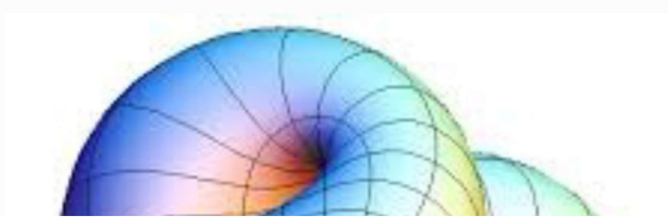
Most generally,

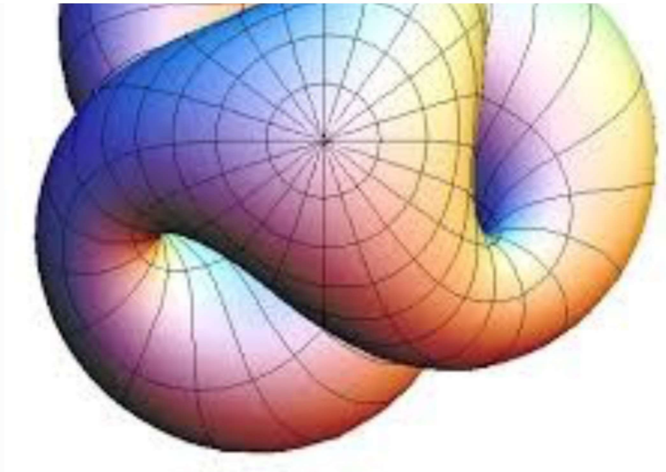
$$\left\{\begin{array}{c}\text{k-dim'l subspaces}\\ \text{in } \mathbb{R}^n\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{k-frame}\\ \text{up to}\\ GL_k(\mathbb{R})\end{array}\right\}$$

$$\updownarrow$$

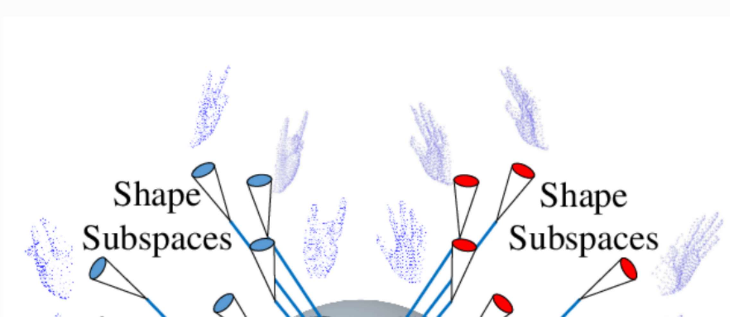
$$\left\{\begin{array}{c}\text{points in}\\ Gr(k,n)\end{array}\right\}$$

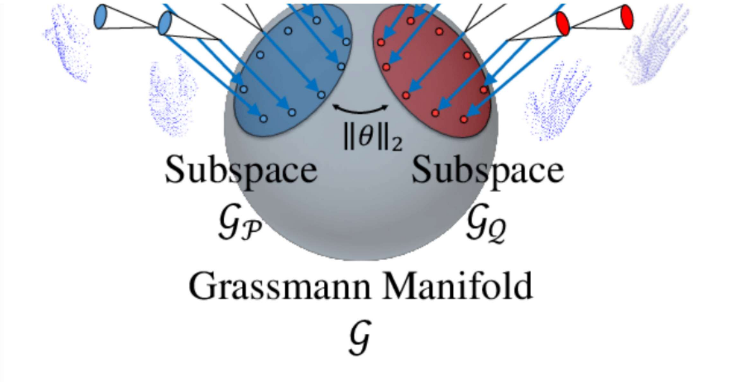
where $Gr(k,n)$ = Grassmannian manifold.

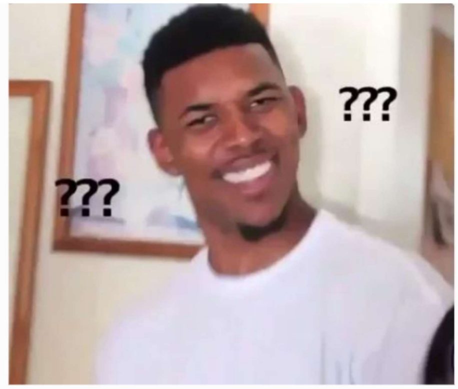




Just kidding, I couldn't find any good pics of a Grassmanian manifold ...





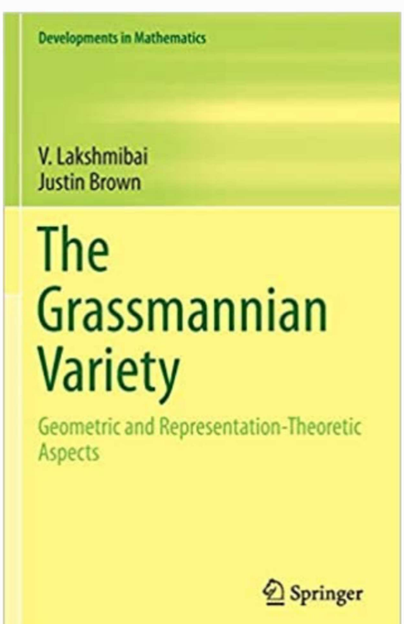


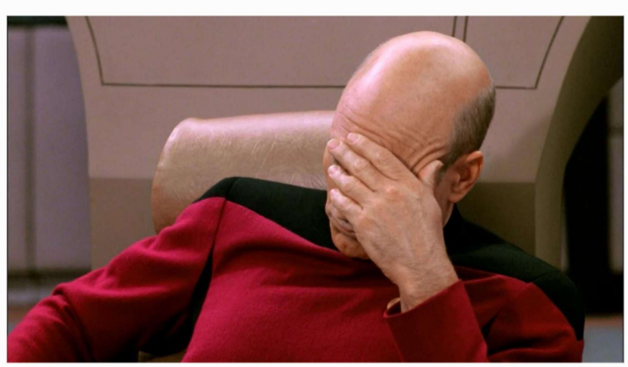
Google searched

"Grassmannian manifold"

Apparently algebraic geometers are

even less imaginative:





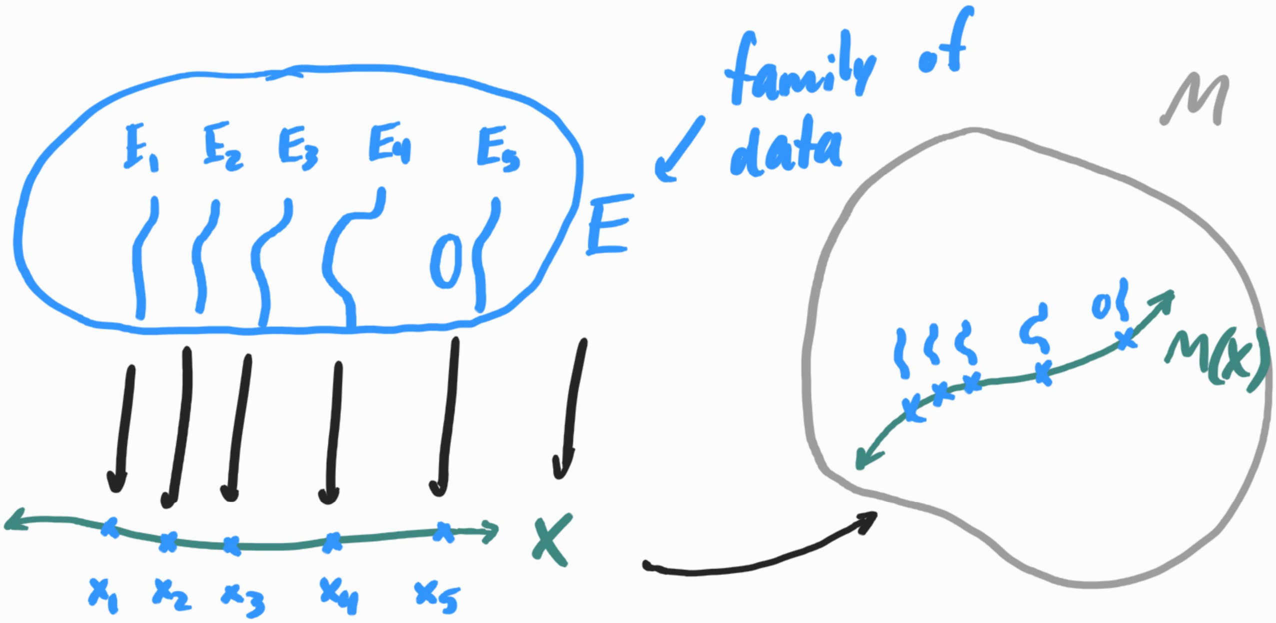
Google searched "Grassmannian variety"

$\boxed{\text{Modern Theory}}$

Let M be a moduli space, so that points of M correspond to some interesting information we'd like to keep track of.

e.g. $M = \mathbb{P}^1 \longleftrightarrow$ lines through the origin in $\mathbb{R}^2$

The real power of M is that it helps us parametrize families of such information:



Slogan: it's all about the arrows.

If $f: Y \longrightarrow X$ is a function

between spaces (that's continuous,

between spaces (...

differentiable, holomorphic, linear,

a homomorphism, etc.) then

we can use f to transfer

information from X over to Y.