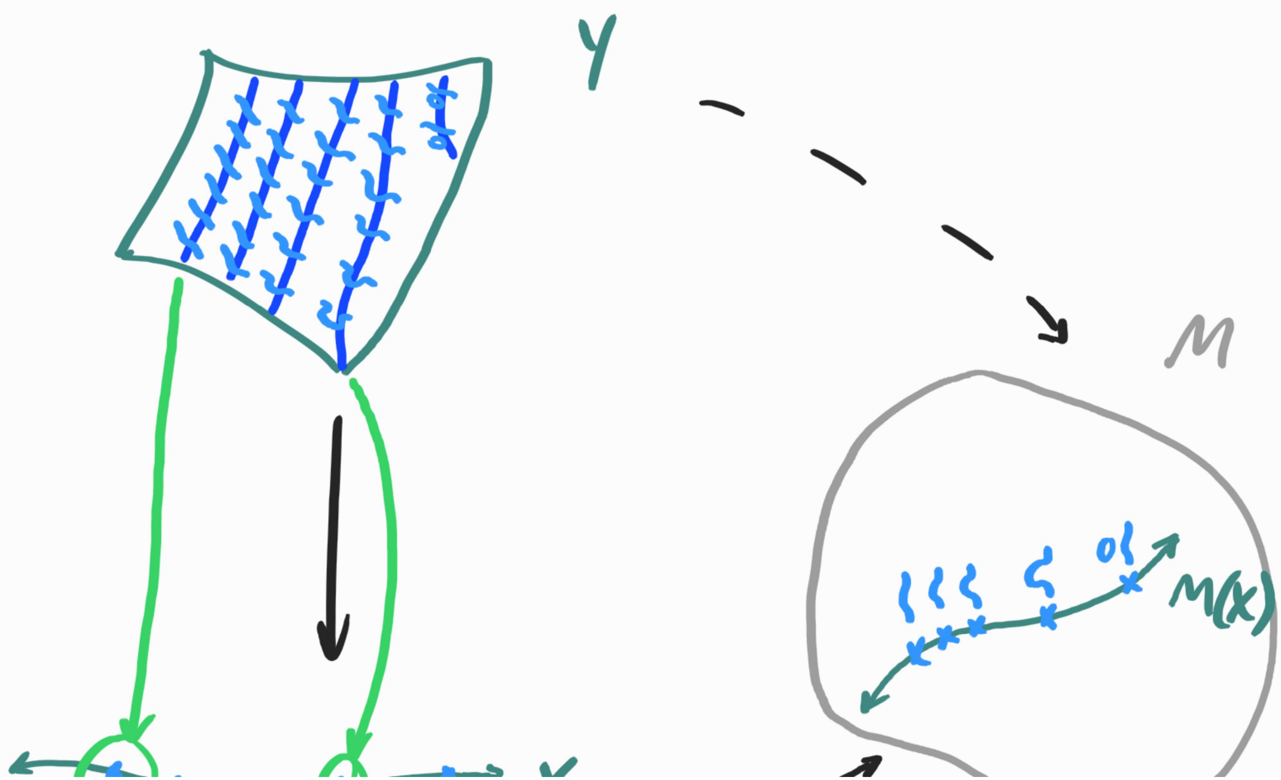
For example, if such information on X comes from mapping X to a moduli space $\mathcal{M}$, then Y also admits a map to $\mathcal{M}$ by composition



Written a different way, a moduli space M encodes:

$$\left\{ \begin{matrix} \text{data} \\ \text{over } X \end{matrix} \right\} \longleftrightarrow \mathrm{Maps}(X, M)$$

data at every pt. of $g(X) \in M$ $\longleftrightarrow$ $g : X \to M$

in a way that is compatible with arrows between Y and X:

$$\left\{ \begin{tikzcd} \text{induced data} \arrow[r] \arrow[d] & \text{data} \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd} \right\} \longleftrightarrow \begin{tikzcd} \mathrm{Maps}(X, M) \arrow[d, "f^*"] \\ \mathrm{Maps}(Y, M). \end{tikzcd}$$

$g : X \to M$

$f^*g : Y \xrightarrow{f} X \xrightarrow{g} M$

For the experts: this says that

$\mathrm{Maps}(-, M)$ is a functor

represented by a space — M!

In general, you can think of a

functor as a particular <u>invariant</u>

of a space

$$F : X \longmapsto F(X) =$$

data over X

e.g. $\dim : V \longmapsto \dim(V)$

vector space

that respects relations between spaces

$$F : \begin{pmatrix} Y \\ \downarrow f \\ X \end{pmatrix} \longmapsto \begin{matrix} F(X) \\ \downarrow f^* \\ F(Y) \end{matrix} .$$

e.g. $\begin{pmatrix} V \\ \rotatebox{90}{$\subseteq$} \\ W \end{pmatrix} \longmapsto \begin{pmatrix} \dim V \\ \rotatebox{90}{$\leq$} \\ \dim W \end{pmatrix}$

In all our examples,

$$F(-) = Maps(-, \mathcal{M})$$

for some moduli space $\mathcal{M}$.

The fun doesn't stop here though...

We could ask for relations between the invariants:

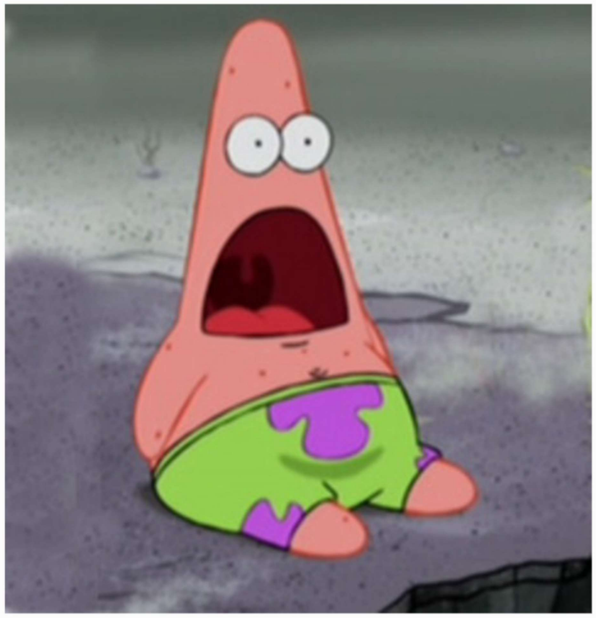
$$F \overset{\eta}{\Rightarrow} G \quad \rightsquigarrow \quad \begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowleft"] & G(x) \arrow[d, "f^*"] \\ F(y) \arrow[r, "\eta_y"'] & G(y) \end{tikzcd}$$

and relations between these relations:

$$\xi \overset{\theta}{\Rrightarrow} \eta$$

$$\begin{tikzcd} F \arrow[r, Rightarrow, "\eta"] & G \\ F \arrow[u, Rightarrow, "\theta_F"] \arrow[r, Rightarrow, "\xi"] & G \arrow[u, Rightarrow, "\theta_G"'] \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd}[row sep=small] & F(x) \arrow[rr, "\eta_x"] \arrow[dd, "f^*"'] \arrow[ddrr, phantom, "\circlearrowleft"] & & G(x) \arrow[dd, "f^*"] \\ F(x) \arrow[ur, "\theta"] \arrow[rr, "\xi_x", near start] \arrow[dd, "f^*"'] \arrow[ddrr, phantom, "\circlearrowleft"] & & G(x) \arrow[ur, "\theta"] \arrow[dd, "f^*"] & \\ & F(y) \arrow[rr, "\eta_y", near start] & & G(y) \\ F(y) \arrow[ur, "\theta"] \arrow[rr, "\xi_y"'] & & G(y) \arrow[ur, "\theta"'] & \end{tikzcd}$$

etc.



Takeaway: treating a problem geometrically — find points on a space M — gives us new tools to solve the problem.

This perspective is at the heart of many areas of modern mathematics, even if it's not referred to in this way.

$\boxed{\text{Thanks for your attention! Questions?}}$