What is a Moduli Problem?

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Poppy

Today's lesson plan:

Introduction

Loose definition: a moduli problem is a problem of the form "classify all objects of a certain type"

usually + "up to equivalence"

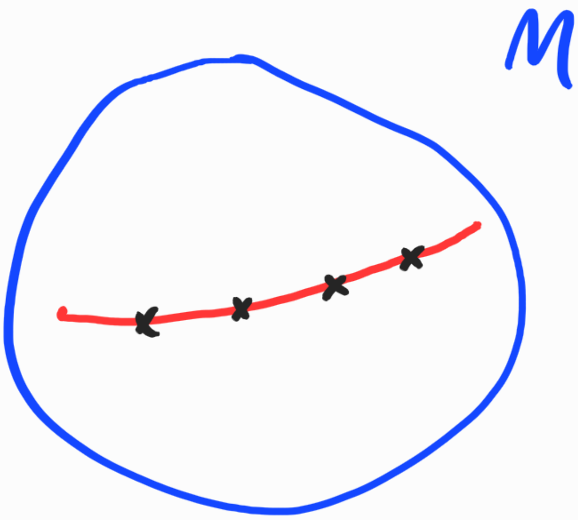
In many situations, the solutions to such problems can be parametrized geometrically — we can view the solutions themselves as points in some parameter space, called a **moduli space**.

The utility of this is:

- if the space M has any extra structure (e.g. a metric, a group structure, the structure of a variety), then this sheds light on all solutions simultaneously.

- most powerfully, we can produce families of solutions by picking meaningful subsets of the moduli space.

family of solutions parametrized by a curve

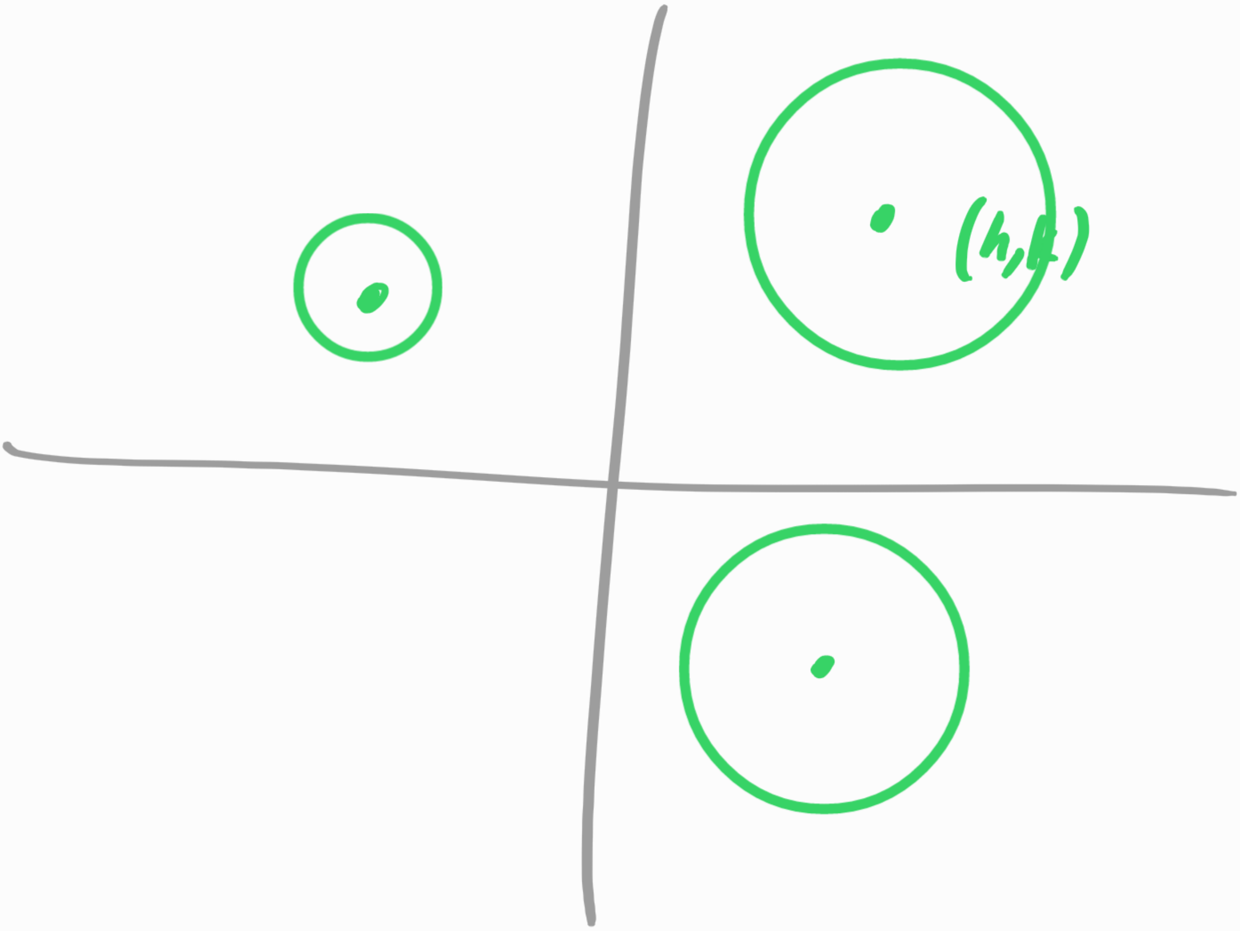


Elementary Examples

Here are some familiar examples:

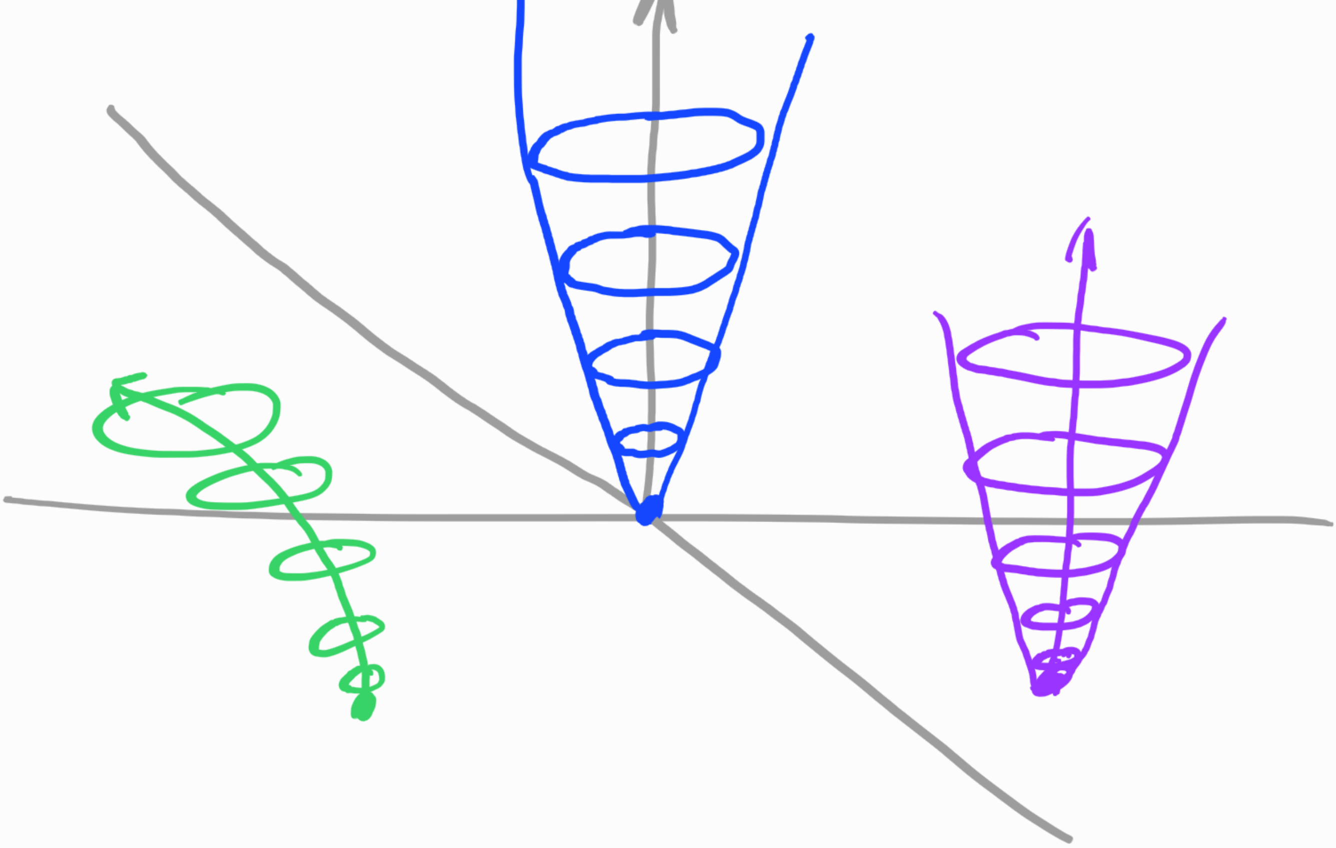
1. Circles in $\mathbb{R}^2$

Every circle $C \subseteq \mathbb{R}^2$ is uniquely specified by <u>center + radius</u>.



A moduli space for circles in $\mathbb{R}^2$

is $\mathbb{R}^2 \times \mathbb{R}_{>0}$ = " $+$ + $\circ\!\!\longrightarrow$ "

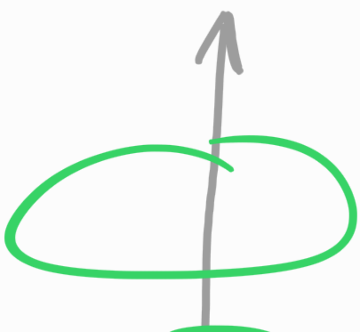


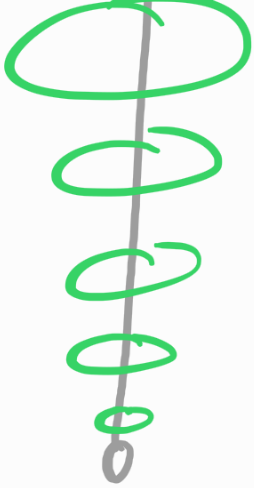
2 Circles in $\mathbb{R}^2$ up to isometry

Two circles $C_1, C_2 \subseteq \mathbb{R}^2$ are **isometric** if they have the same <u>size</u>.

So circles up to isometry are classified by radius.

A moduli space for circles up to isometry is $\mathbb{R}_{>0}$





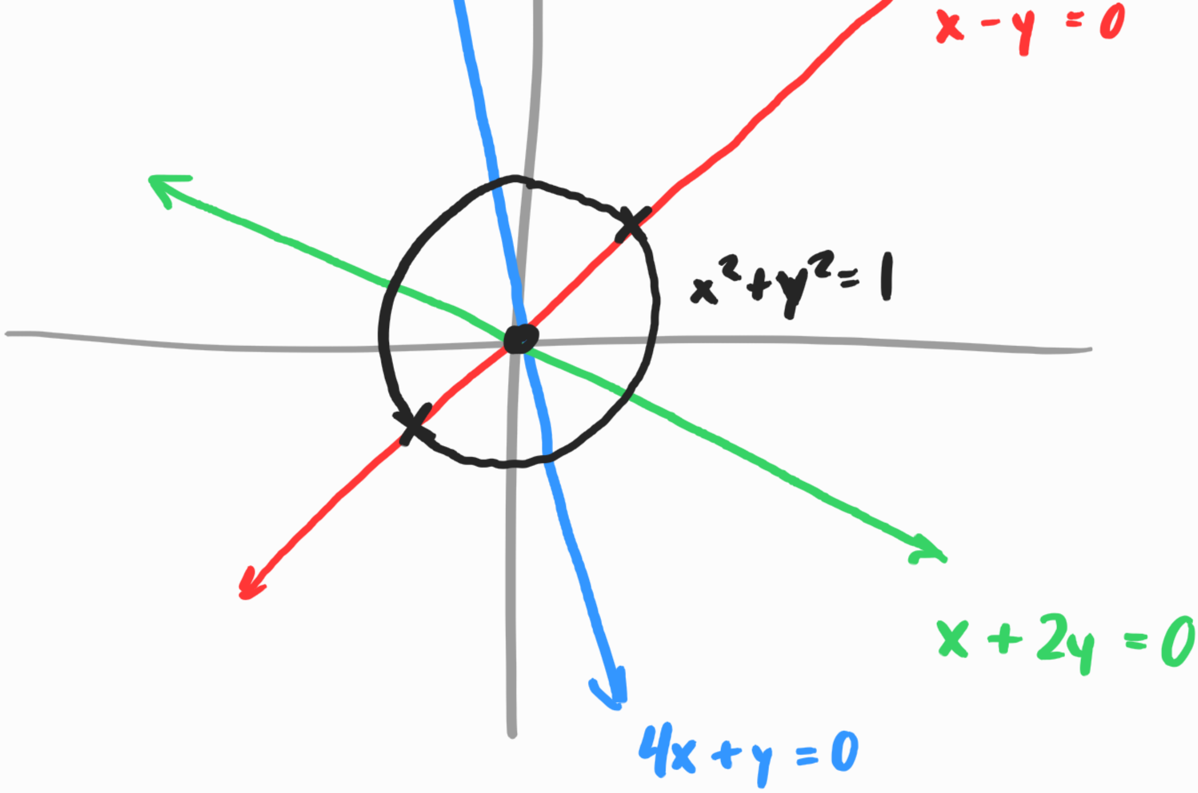
③ Lines in $\mathbb{R}^2$

A line in $\mathbb{R}^2$ is given by an equation $ax + by = c$, so we can use $\mathbb{R}^3 = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}$

BUT this misses when two equations represent the same line

e.g. $(x + y = 1) \simeq (2x + 2y = 2)$

To make things simpler, let's just consider lines through $(0,0)$:



$L: \quad ax + by = 0$

A line through the origin is <u>uniquely</u> specified by

slope $= \frac{-b}{a}$ (allow $\infty = \frac{1}{0}$)

OR

a point on C up to

antipodes

A moduli space of lines through the origin is

$\mathbb{P}^1$ = the projective line

= { the set of slopes of lines through $(0,0)$ }

$$= C/\sim$$

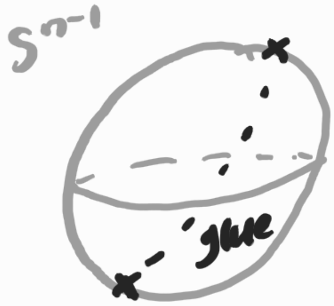
④ More generally,

$$\left\{\begin{matrix}\text{lines through} \\ \vec{0} \text{ in } \mathbb{R}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{unit vector} \\ \text{up to } \pm 1\end{matrix}\right\}$$

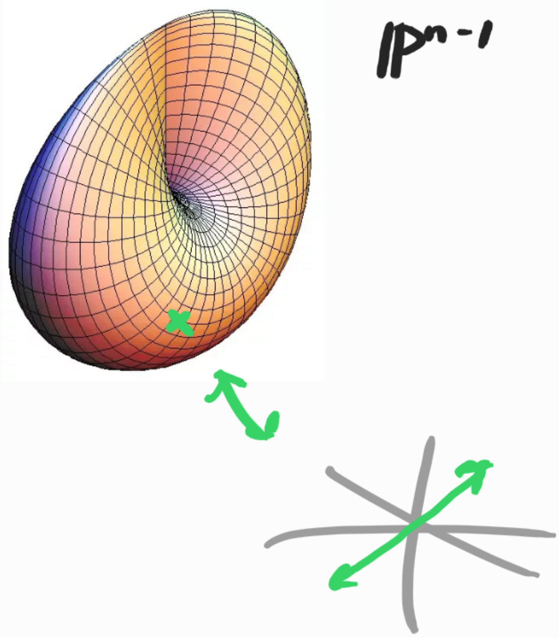
$$\updownarrow$$

$$\left\{\begin{matrix}\text{points in} \\ \mathbb{P}^{n-1}\end{matrix}\right\}$$

where $\mathbb{P}^{n-1} = S^{n-1} / \text{identify antipodes}$



$\rightsquigarrow$

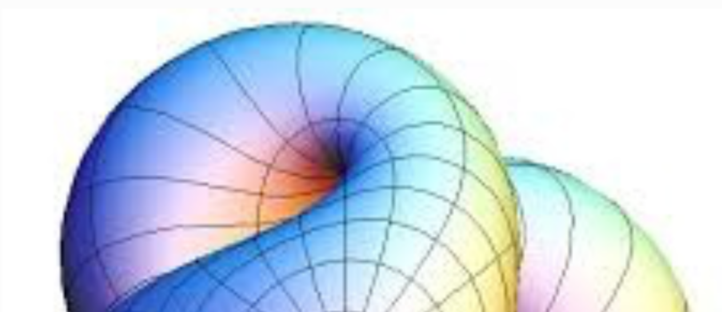


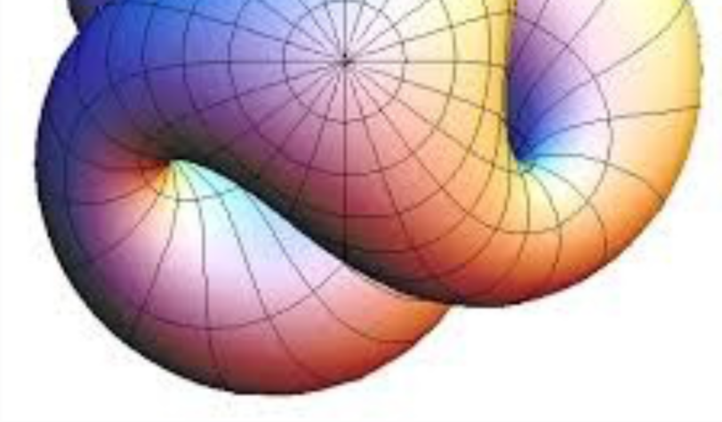
$\mathbb{P}^{n-1}$

Most generally,

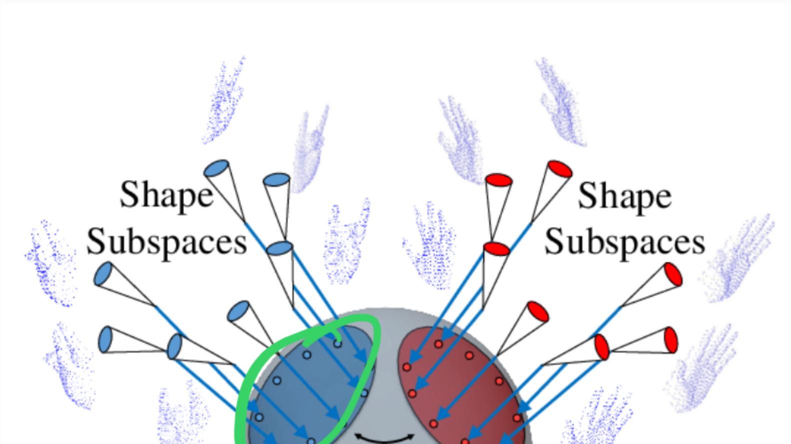
$$\begin{tikzcd}
\left\{\begin{matrix}k\text{-dim'l subspaces}\\ \text{in } \mathbb{R}^n\end{matrix}\right\} \arrow[r, leftrightarrow] & \left\{\begin{matrix}k\text{-frame}\\ \text{up to}\\ GL_k(\mathbb{R})\end{matrix}\right\} \arrow[d, leftrightarrow] \\
 & \left\{\begin{matrix}\text{points in}\\ Gr(k,n)\end{matrix}\right\}
\end{tikzcd}$$

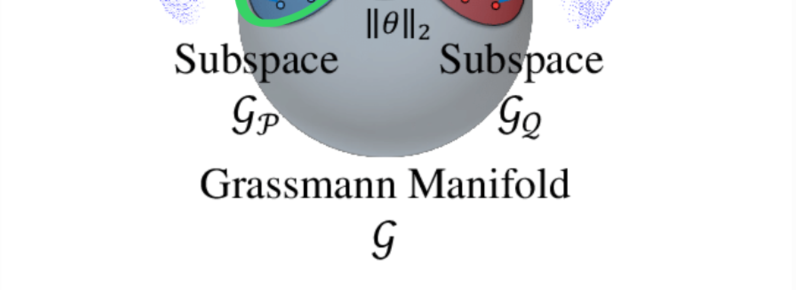
where $Gr(k,n)$ = Grassmannian manifold.

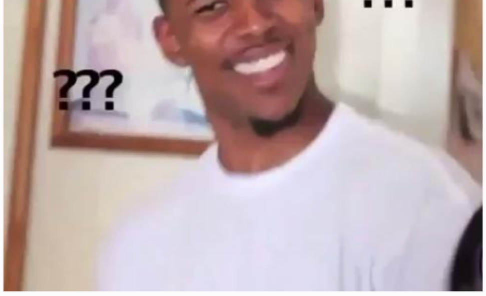




Just kidding, I couldn't find any good pics of a Grassmanian manifold ...

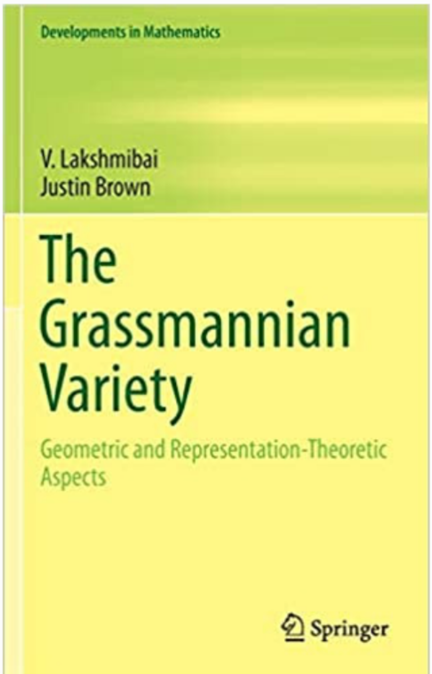


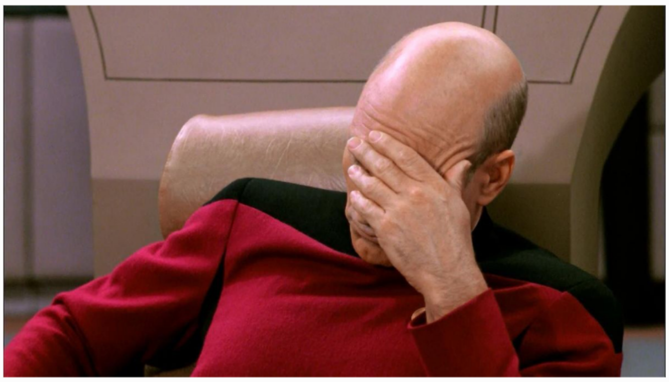




Google searched
"Grassmannian manifold"

Apparently algebraic geometers are
even less imaginative :





Google searched

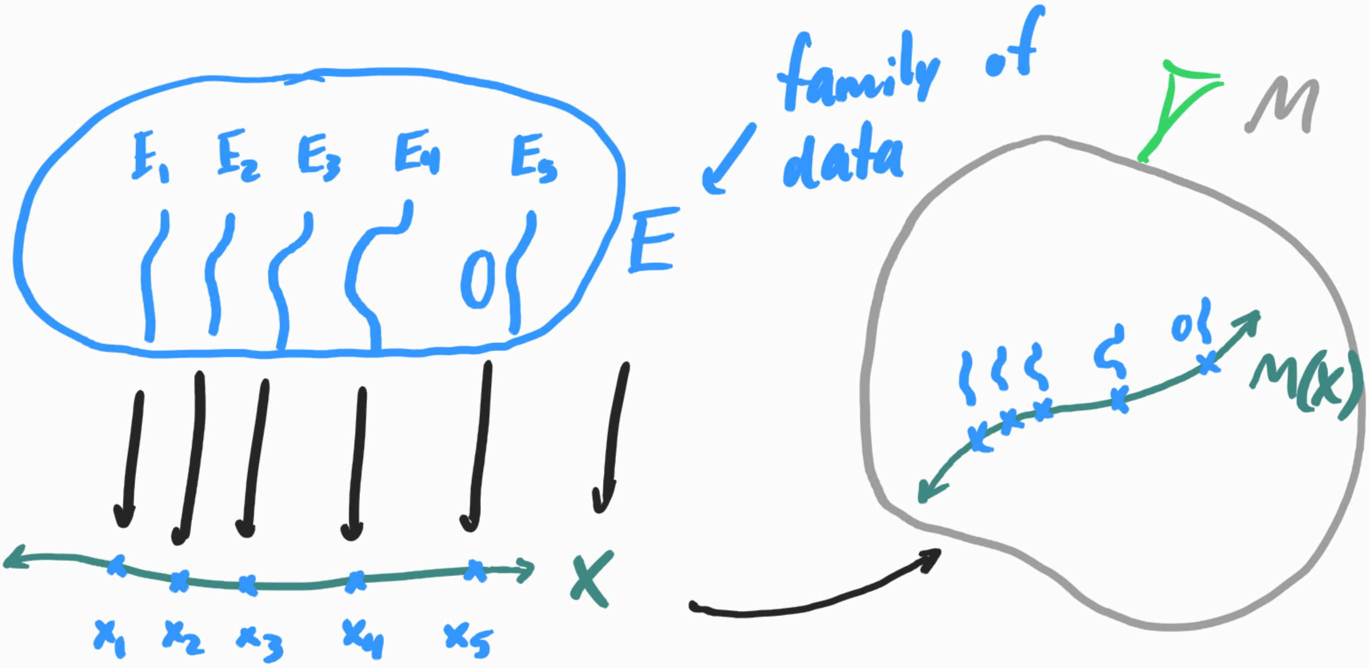
"Grassmannian variety"

Modern Theory

Let M be a moduli space, so that points of M correspond to some interesting information we'd like to keep track of.

e.g. $M = \mathbb{P}^1 \longleftrightarrow$ lines through the origin in $\mathbb{R}^2$

The real power of M is that it helps us parametrize families of such information:

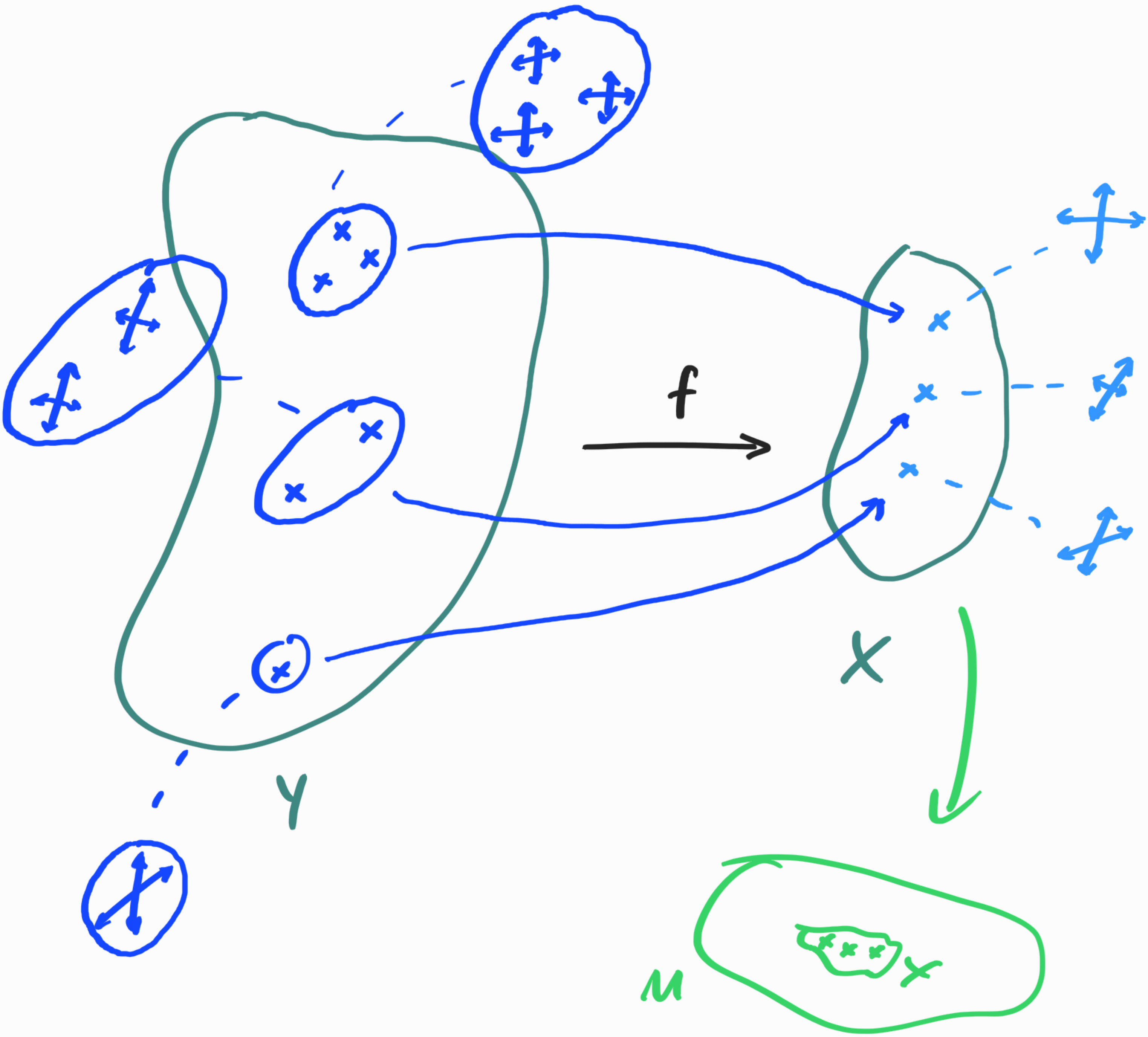


Slogan: it's all about the arrows.

If $f: Y \longrightarrow X$ is a function

between spaces (that's continuous,

differentiable, holomorphic, linear, a homomorphism, etc.) then we can use f to transfer information from X over to Y.



For example, if such information on X comes from mapping X to a moduli space M, then Y also admits a map to M by composition