Written a different way, a moduli space M encodes:

$$\left\{\begin{array}{c}\text{data}\\ \text{over } X\end{array}\right\} \longleftrightarrow \mathrm{Maps}(X, M)$$

$$\begin{array}{c}\text{1 piece of}\\ \text{data}\end{array} \longleftrightarrow \ast \longrightarrow M$$

in a way that is compatible with arrows between Y and X:

$$\left\{\begin{tikzcd} \text{induced data} \arrow[r] \arrow[d] & \text{data} \arrow[d] \\ Y \arrow[r, "f"] & X \end{tikzcd}\right\} \longleftrightarrow \begin{tikzcd} \mathrm{Maps}(X, M) \arrow[d, "f^*"] \\ \mathrm{Maps}(Y, M). \end{tikzcd}$$

$$X \to M \overset{f^*}{\rightsquigarrow} Y \xrightarrow{f} X \to M$$

For the experts: this says that

$\mathrm{Maps}(-, M)$ is a functor

represented by a space — M!

In general, you can think of a

functor as a particular invariant

of a space

$$F : X \longmapsto F(X) = $$

data over X

e.g. $\dim : V \longmapsto \dim(V)$

vector space

that respects relations between

spaces

$$F : \begin{pmatrix} Y \\ \downarrow f \\ X \end{pmatrix} \longmapsto \begin{matrix} F(X) \\ \downarrow f^* \\ F(Y) \end{matrix} .$$

e.g. $\begin{pmatrix} V \\ \uparrow \\ W \end{pmatrix} \longmapsto \begin{pmatrix} \dim V \\ \geq \\ \dim W \end{pmatrix}$

In all our examples,

$$F(-) = Maps(-, M)$$

for some moduli space M.

The fun doesn't stop here though...

We could ask for relations between

the invariants:

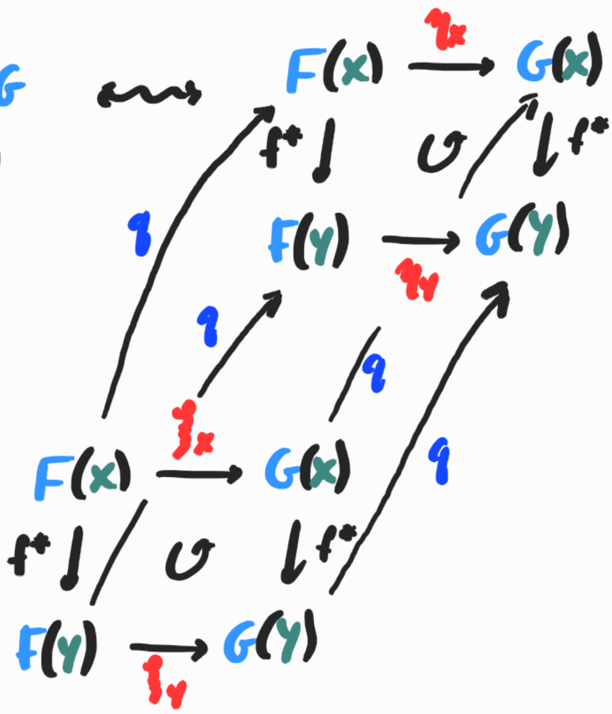
$$F \overset{\eta}{\Rightarrow} G \quad \rightsquigarrow \quad \begin{tikzcd} F(x) \arrow[r, "\eta_x"] \arrow[d, "f^*"'] \arrow[dr, phantom, "\circlearrowleft"] & G(x) \arrow[d, "f^*"] \\ F(y) \arrow[r, "\eta_y"'] & G(y) \end{tikzcd}$$

\# sol'ns to $Ax=b$ $\rightsquigarrow$ dim ker

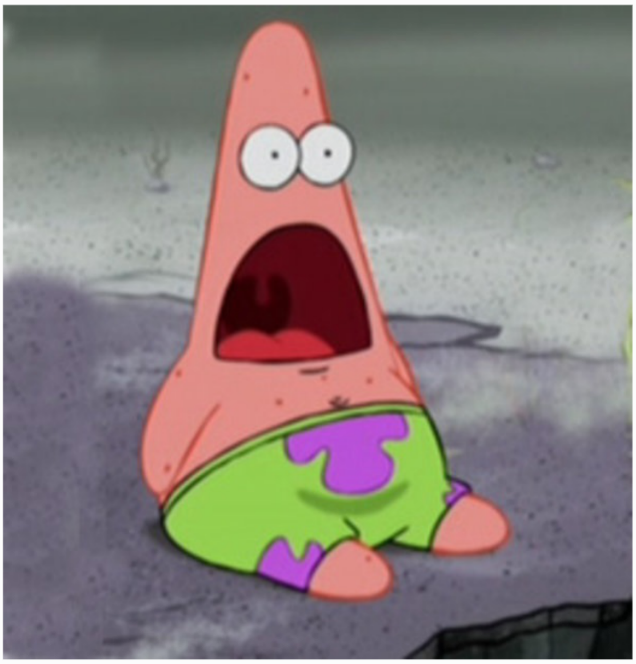
and relations between these relations:

$$\xi \overset{q}{\Rrightarrow} \eta$$

$$\begin{tikzcd} & F \arrow[r, Rightarrow, "\eta"] & G \\ F \arrow[r, Rightarrow, "\xi"'] \arrow[ur, Rightarrow, "q_F"] & G \arrow[ur, Rightarrow, "q_G"'] & \end{tikzcd}$$



etc.



Takeaway: treating a problem geometrically — find points on a space M — gives us new tools to solve the problem.

This perspective is at the heart

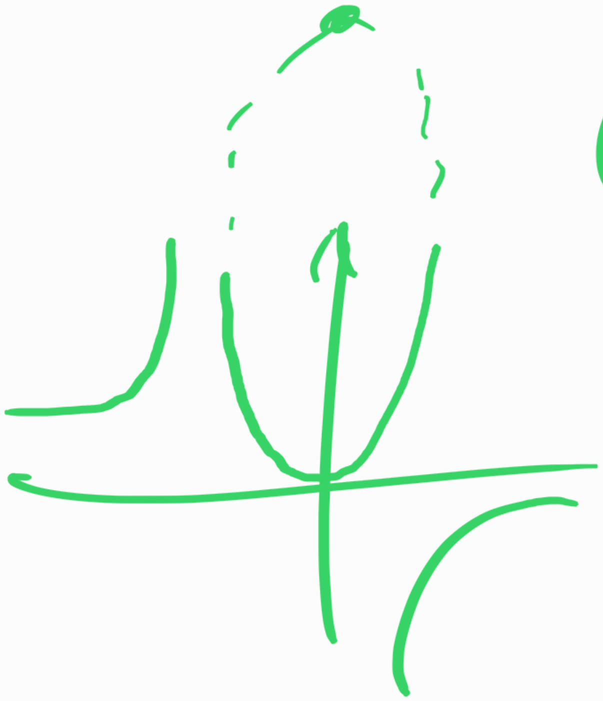
This perspective is at the heart of many areas of modern mathematics, even if it's not referred to in this way.

Category theory

$\boxed{\text{Thanks for your attention! Questions?}}$

$\{\text{Conics}\} \longleftrightarrow ax^2 + bxy + cy^2 +$

$dx + ey + f = 0$



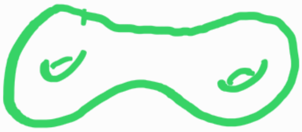
$(a, b, \ldots, f)$

$\{\text{conics}/\text{homeo.}\} = * =$

$\{\bigcirc\}$

"Parametrize algebraic things algebraically"

{ hyp. structures on



genus $g \geq 2$ } $\longleftrightarrow$ $6g-6$ coord's (Fenchel-Nielsen)

$\updownarrow$

$\mathbb{R}^{6g-6}$

HypStr