What is ... a Moduli Space?

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Begonia

Poppy

Today's lesson plan:

$\boxed{\text{Introduction}}$

loose definition: a moduli problem is a problem of the form

"classify all objects of a certain type"
"

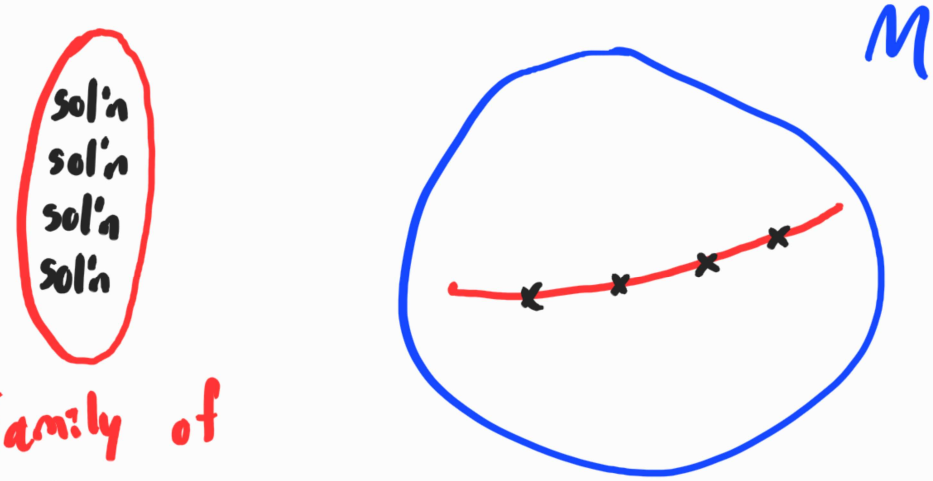
usually + "up to isomorphism"

In many situations, the solutions to such problems can be parametrized geometrically — we can view the solutions themselves as points in some parameter space, called a **moduli space**.

The utility of this is:

- if the space M has any extra structure (e.g. a metric, a group structure, the structure of a variety), then this sheds light on all solutions simultaneously.

- most powerfully, we can produce families of solutions by picking meaningful subsets of the moduli space.



family of solutions parametrized by a curve

Here are some familiar examples:

① Circles in $\mathbb{R}^2$

Every circle $C \subseteq \mathbb{R}^2$ is uniquely specified by <u>center + radius</u>.

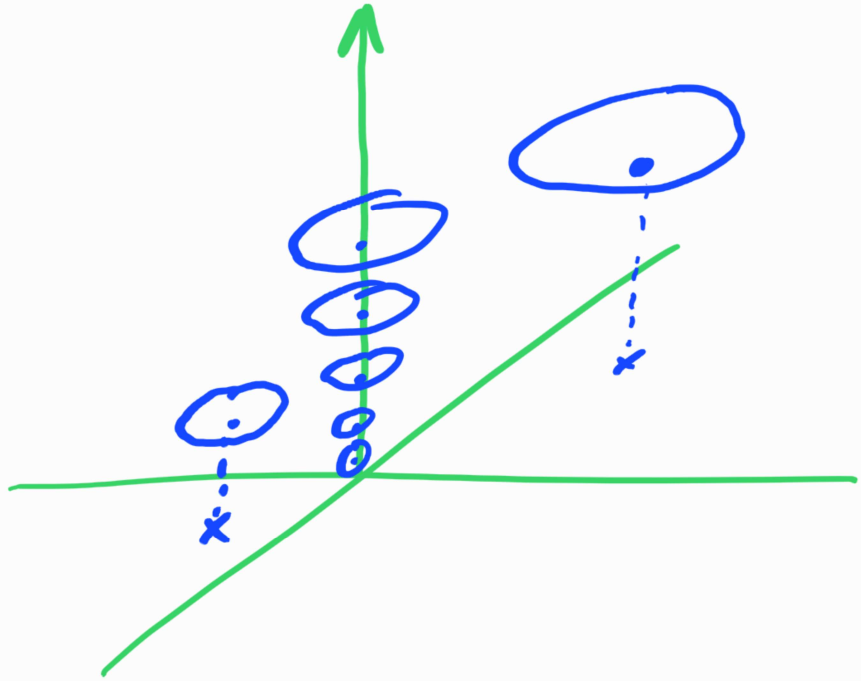
$(x,y) \in \mathbb{R}^2$

$r \in (0, \infty)$

A moduli space for circles in $\mathbb{R}^2$

is

$$\mathbb{R}^2 \times \mathbb{R}_{>0}$$

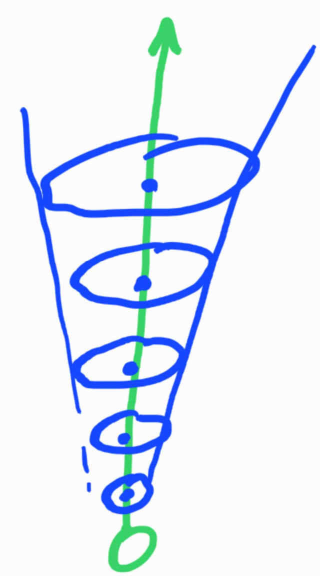


② Circles in $\mathbb{R}^2$ up to isometry

Two circles $C_1, C_2 \subseteq \mathbb{R}^2$ are isometric if they have the same <u>size</u>.

So circles up to isometry are classified by ___radius___.

A moduli space for circles up to isometry is $\mathbb{R}_{>0}$



③ Lines in $\mathbb{R}^2$

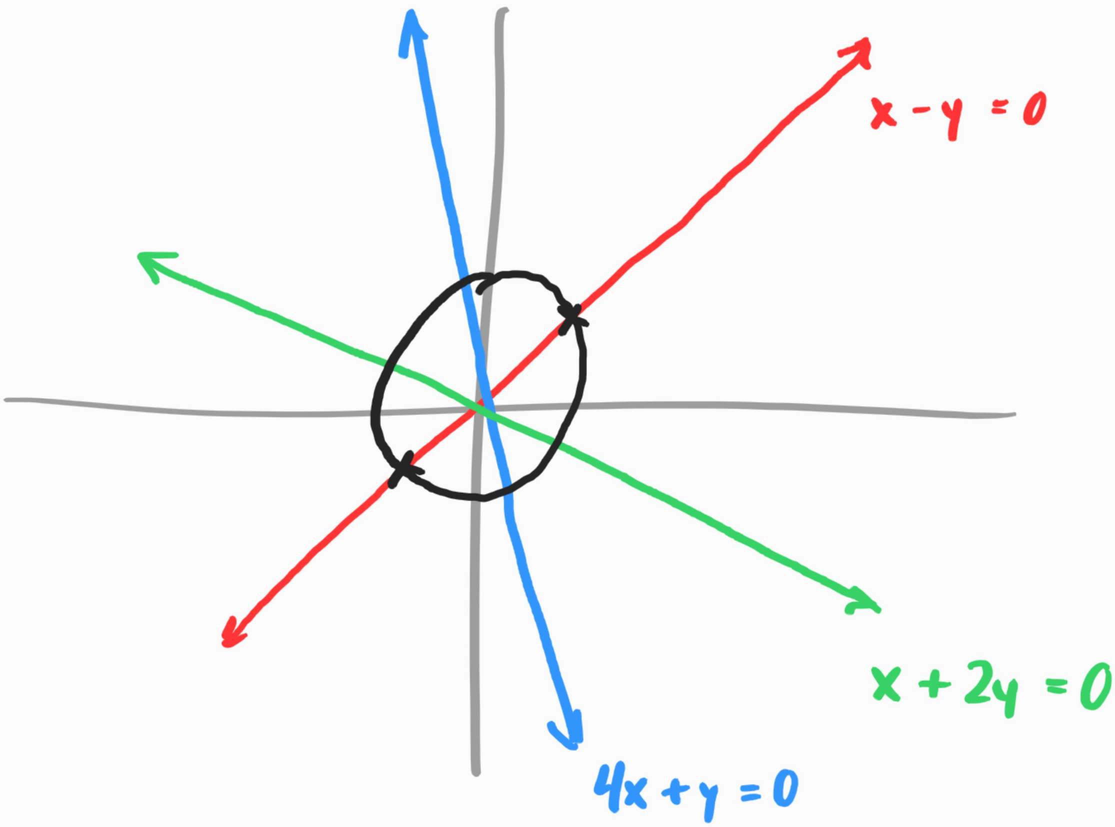
A line in $\mathbb{R}^2$ is given by an equation $ax + by = c$, so we can think of

$$\mathbb{R}^3 = \{(a,b,c) \mid a,b,c \in \mathbb{R}\}.$$

BUT this misses when two equations represent the same line

e.g. $(x + y = 1) \simeq (2x + 2y = 2)$

To make things simpler, let's just consider lines through (0,0):



$$L: \quad ax + by = 0$$

A line through the origin is <u>uniquely</u> specified by

✓ slope $= \frac{-a}{b} \in \mathbb{R}$

✓ <u>any point</u> in $\mathbb{R}^2 \setminus \{0\}$ up to scaling $(\mathbb{R}^\times)$

✓ a point on $C = \{x^2+y^2=1\}$ up to ± 1

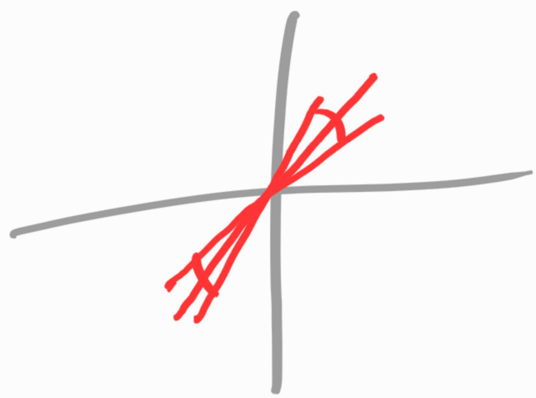
A moduli space of lines through the origin is

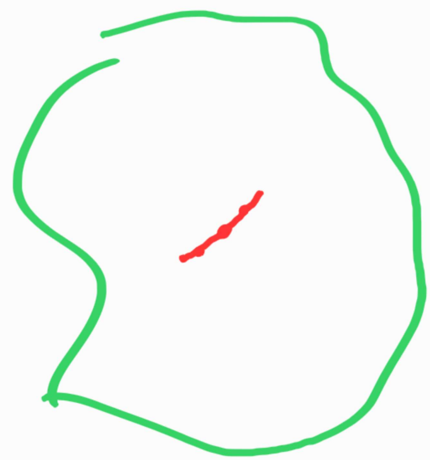
$\mathbb{P}^1 = C/\sim$ where

$x \sim -x$

"projective line"

$\mathbb{P}^1$





④ More generally,

$$\left\{\begin{matrix}\text{lines through} \\ \vec{0} \text{ in } \mathbb{R}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{unit vector} \\ \text{up to } \pm 1\end{matrix}\right\}$$

$$\Big\updownarrow$$

$$\left\{\begin{matrix}\text{points in}\\ \mathbb{P}^{n-1}\end{matrix}\right\}$$

where $\mathbb{P}^{n-1} = S^{n-1} \Big/ \begin{matrix}\text{identify}\\ \text{antipodes}\end{matrix}$

Most generally,

$$\left\{\begin{matrix}\text{k-dim'l subspaces}\\ \text{in } \mathbb{R}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{k-frame}\\ \text{up to}\\ GL_k(\mathbb{R})\end{matrix}\right\}$$

$$\Big\updownarrow$$

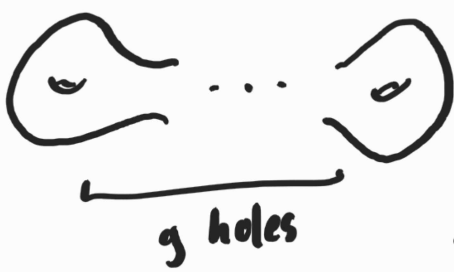
$$\left\{\begin{matrix}\text{points in}\\ Gr(k,n)\end{matrix}\right\}$$

where $Gr(k,n)$ = Grassmannian manifold.

Here are some more complicated ones:

5 Complex/hyperbolic surfaces

$\left\{ \begin{array}{c} \text{complex/hyperbolic} \\ \text{structures on} \end{array} \right.$



$\left. \right\} \longleftrightarrow \left\{ \text{Fenchel-Nielsen coordinates} \right\}$

$g \geq 2$

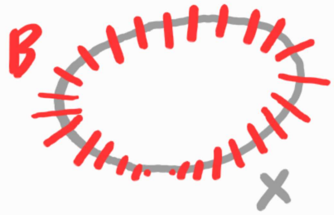
$\updownarrow$

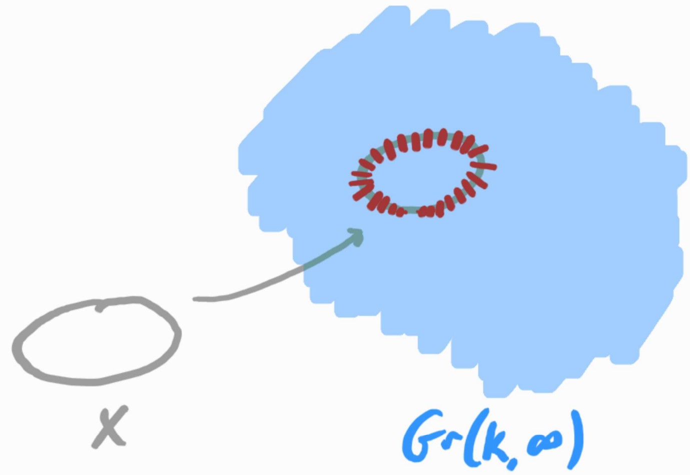
$\mathbb{R}^{6g-6}$

6 Vector bundles and G-bundles

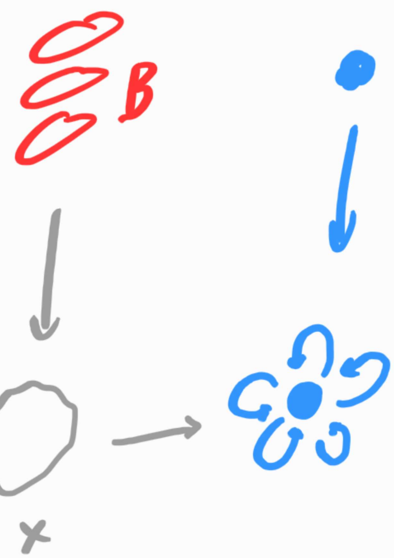
(Rank k vector)

$$\left\{ \begin{array}{c} \text{bundles on } X \\ \text{up to iso.} \end{array} \right\} \longleftrightarrow Gr(k, \infty)$$





$$\left\{ \begin{array}{c} \text{principal} \\ G\text{-bundles} \\ \text{up to iso.} \end{array} \right\} \longleftrightarrow BG$$



Lines on Cubic Surfaces