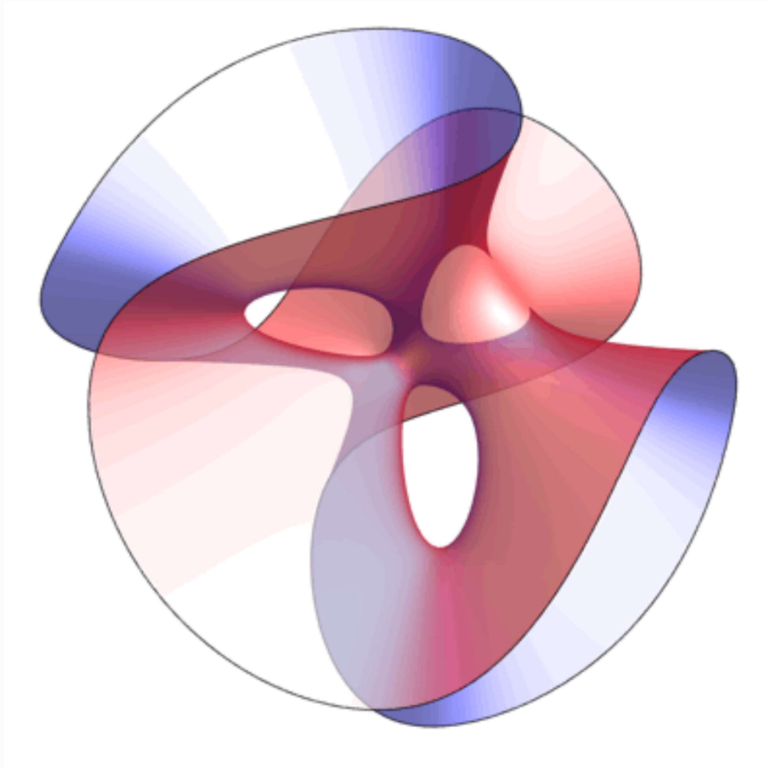
In algebraic geometry, moduli spaces are used to parametrize families of algebraic varieties.

Slogan: parametrize algebraic objects algebraically.

A cubic surface is the set of solutions to a degree 3 (homogeneous) equation in 4 variables.

Ex The Clebsch surface



$$x^3+y^3+z^3+w^3=(x+y+z+w)^3$$

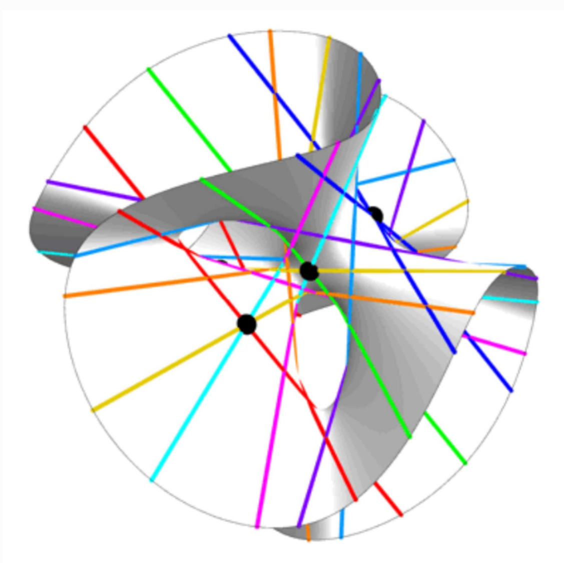
Such a surface is **smooth** if it has no singularities.

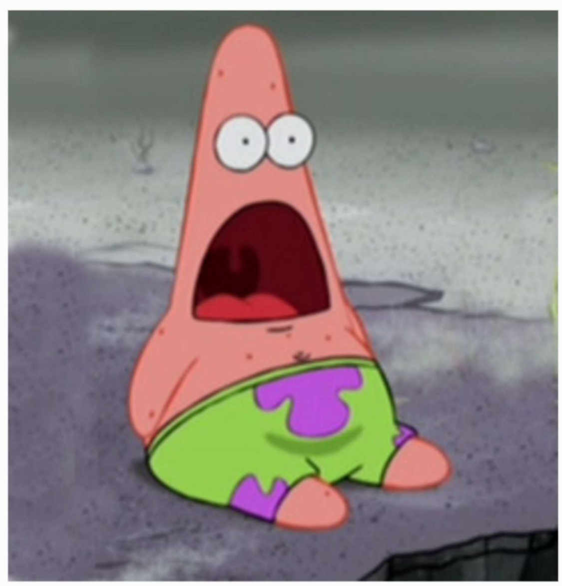
e.g. $x^3=0$ is singular

Theorem (Cayley 1849) A smooth

cubic surface contains exactly 27 lines.

That is, the moduli space of "lines on a smooth cubic" is a 0-dimensional variety with 27 components.





Today, I want to explain how

to use moduli spaces to attack

the problem.

$= \text{lines in } \mathbb{C}^4$

Key insight: lines in $\mathbb{P}^3$ correspond

$\cup\!\!\mid$ $\mathbb{C}^3$

to points on the Grassmannian

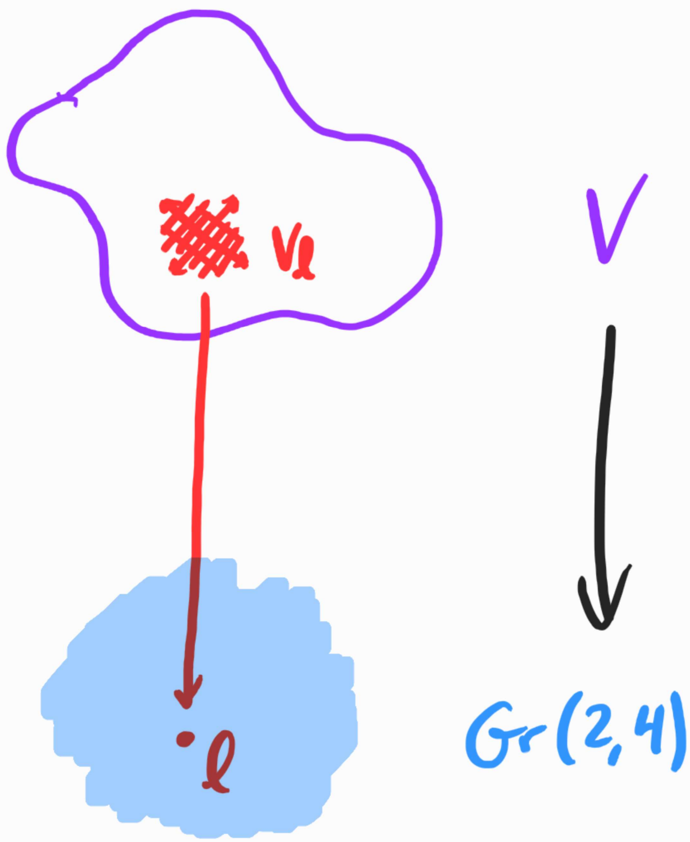
$$Gr(2,4) = \left\{ \begin{matrix} 2\text{-dim'l subspaces} \\ \text{in } \mathbb{C}^4 \end{matrix} \right\}.$$

Strategy: view lines $\ell \subseteq \mathbb{P}^3$ as cohomology classes $[\ell] \in H^*(Gr(2,4))$.

There is a rank 2 vector bundle

$$V \longrightarrow Gr(2,4)$$

whose fibre over a line $\ell \subseteq \mathbb{P}^3$ is the space of homogeneous linear forms on ℓ:



Taking the symmetric power

$$E := \mathrm{Sym}^3 V \longrightarrow Gr(2,4)$$

yields a rank 4 vector bundle whose fibres are

$$E_\ell = \{\text{homogeneous cubic forms on } \ell\}.$$

For a cubic surface

$$S = \{F(x,y,z,w) = 0\}$$

let $s_F : Gr(2,4) \longrightarrow E$ be the section $\ell \longmapsto F|_\ell$.

Then $\{\text{lines on } S\} \longleftrightarrow \{\text{zeroes of } s_F\}$.

To count these zeroes, take the

Chern class

$$c_4(E) = \sum_{s_F(\ell)=0} [\ell] \in H^8(Gr(2,4))$$

and do some Schubert calculus:

$$c_4(E) = 9c_1(V)(2c_1(V)^2 + c_2(V))$$

$$= 27[\ell].$$

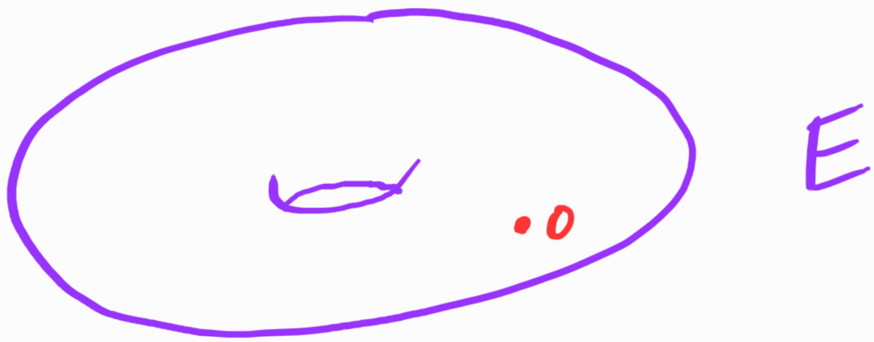
Takeaway: counting problems can be solved with moduli spaces.

$\boxed{\text{Elliptic Curves}}$

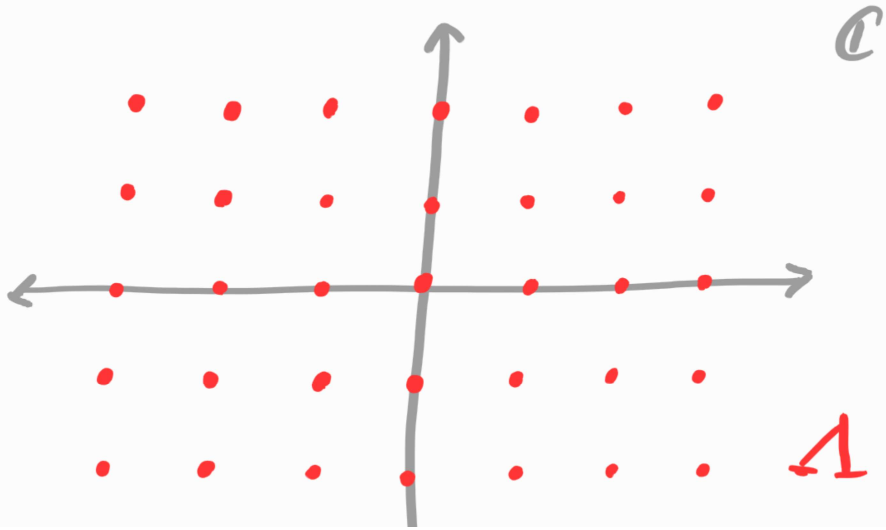
A complex elliptic curve is a

smooth, projective 1-dimensional

complex variety of genus 1, with a specified basepoint.



Elliptic curves are formed by taking a lattice in $\mathbb{C}$



and forming the quotient

$$E = \mathbb{C}/\Lambda.$$

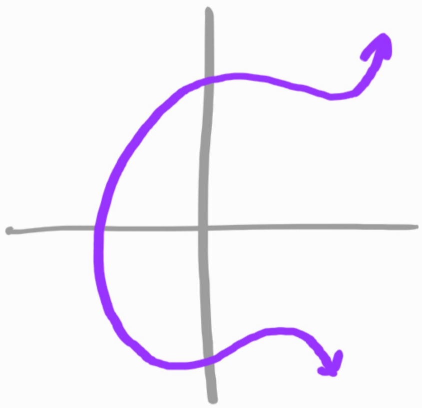
Useful fact:

Every elliptic curve is given by a cubic equation called a **Weierstrass equation**:

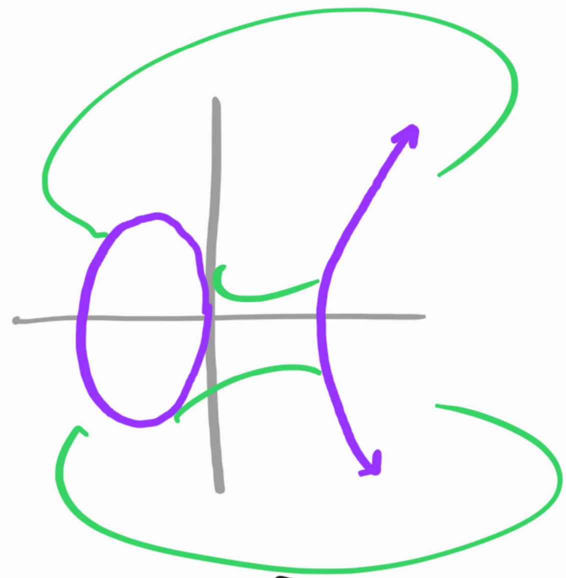
$$E: \ y^2 = x^3 + Ax + B$$

$A, B \in \mathbb{C}$.

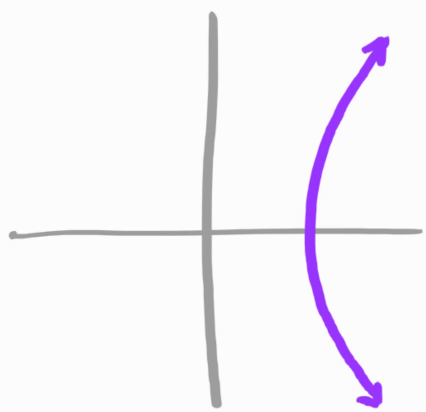
$\mathbb{R}$-points:



$y^2 = x^3 - 3x + 3$



$y^2 = x^3 - x$



$y^2 = x^3 - 1$

For an elliptic curve $E: y^2 = x^3 + Ax + B$,

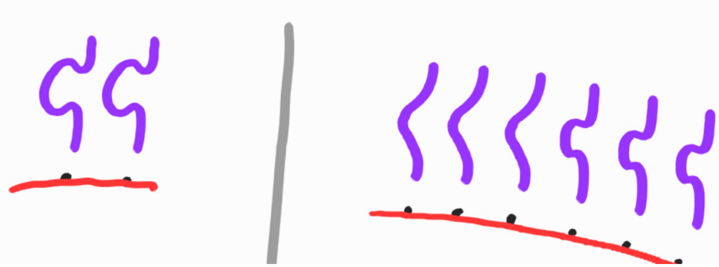
the j-invariant of E is

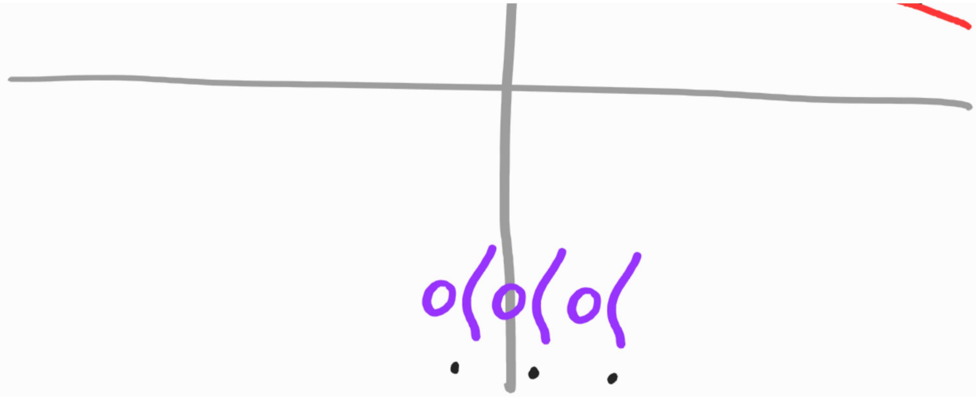
$$j(E) = 1728\,\frac{4A^3}{4A^3+27B^2} \in \mathbb{C}.$$

Theorem Two elliptic curves E, E' are isomorphic if and only if $j(E) = j(E')$.

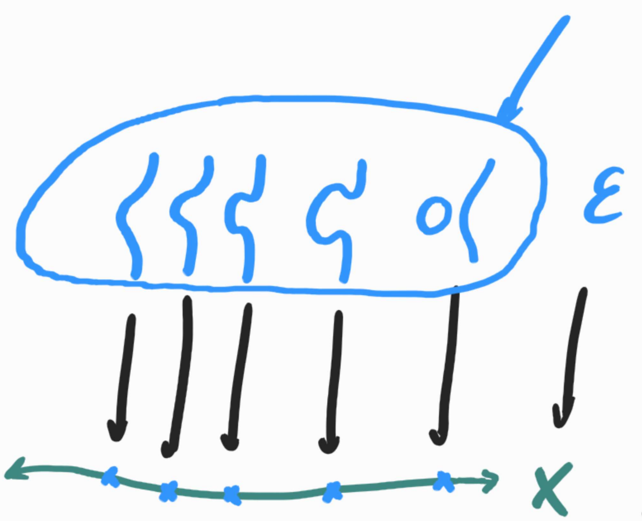
Corollary The complex plane is a moduli space for the isomorphism classes of complex elliptic curves.

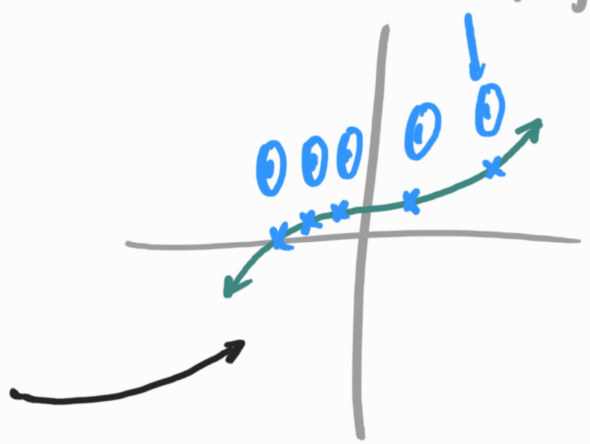
$\mathbb{A}^1_j := \mathbb{C}$

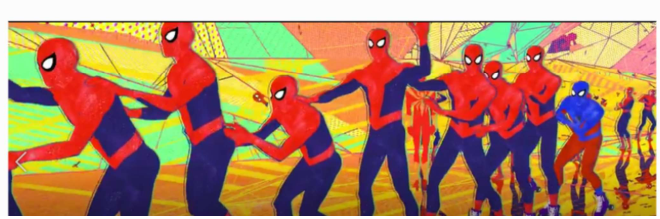


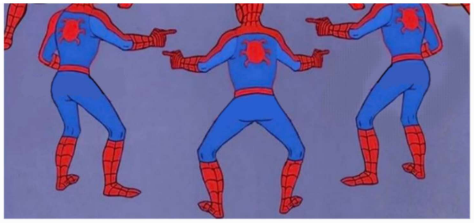


The real power of $\mathbb{A}^1_j$ is that it helps us parametrize families of elliptic curves: $\overline{\mathcal{M}}_{1,1} = \mathbb{P}^1_j \supseteq \mathbb{A}^1_j$









Modern story: it's all about the __arrows__!

$$\begin{tikzcd} \mathcal{E}' \arrow[r, dashed] \arrow[d, dashed] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

A moduli problem is really a functor $\mathcal{M} : \mathrm{Var}_{\mathbb{C}} \longrightarrow \mathrm{Set}$

$$X \longmapsto \mathcal{M}(X)$$

$$\begin{tikzcd} Y \arrow[d, "f"'] \\ X \end{tikzcd} \longmapsto \begin{tikzcd} \mathcal{M}(Y) \\ \mathcal{M}(X) \arrow[u, "f^*"'] \end{tikzcd}$$

For $\mathcal{M}_{1,1} : X \longmapsto \mathcal{M}_{1,1}(X) = \left\{ \begin{matrix} \text{elliptic curves} \\ \mathcal{E} \to X \end{matrix} \right\} / \text{iso.}$

the functor is representable*:

$$\mathcal{M}_{1,1} \cong \mathrm{Hom}(-, \mathbb{A}_j^1)$$

More moduli spaces to study:

- higher genus curves $\mathcal{M}_g$

  (and their compactifications $\overline{\mathcal{M}_g}$)

- elliptic curves with level structure

$Y(N)$, $Y_0(N)$, $Y_1(N)$

$\left(\text{and their compactifications } X(N),\ X_0(N),\ X_1(N)\right)$

- abelian varieties $\mathcal{A}_g$
- K3 surfaces, Calabi-Yau varieties, etc.

$\boxed{\text{Thanks for your attention! Questions?}}$