What is ... a Moduli Space?

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Begonia

Poppy

Today's lesson plan:

Introduction

Loose definition: a **moduli problem** is a problem of the form

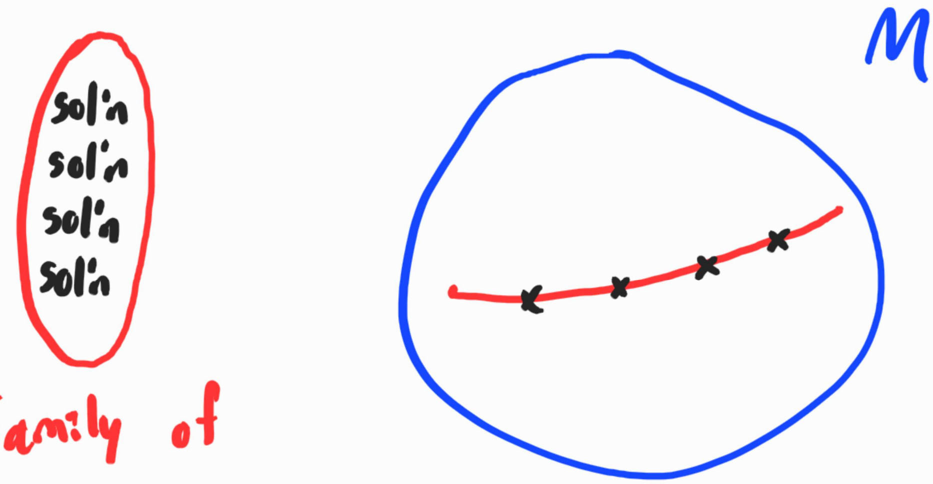
"classify all objects of a certain type" usually + "up to isomorphism"

In many situations, the solutions to such problems can be parametrized geometrically — we can view the solutions themselves as points in some parameter space, called a **moduli space**.

The utility of this is:

- if the space M has any extra structure (e.g. a metric, a group structure, the structure of a variety), then this sheds light on all solutions simultaneously.

- most powerfully, we can produce families of solutions by picking meaningful subsets of the moduli space.



family of solutions parametrized by a curve

Here are some familiar examples:

1. Circles in $\mathbb{R}^2$

Every circle $C \subseteq \mathbb{R}^2$ is uniquely specified by ____________________.

A moduli space for circles in $\mathbb{R}^2$

is

2 Circles in $\mathbb{R}^2$ up to isometry

Two circles $C_1, C_2 \subseteq \mathbb{R}^2$ are isometric if they have the same <u>size</u>.

So circles up to isometry are classified by ________.

A moduli space for circles up to isometry is

③ Lines in $\mathbb{R}^2$

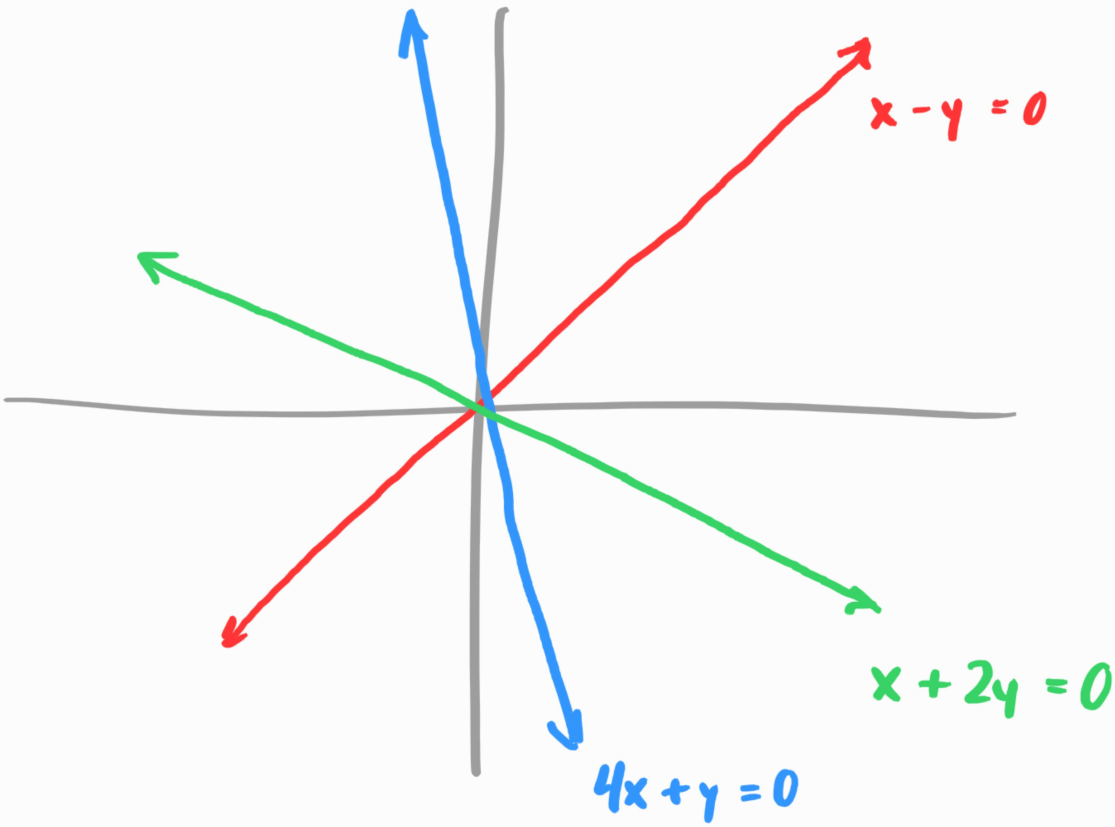
A line in $\mathbb{R}^2$ is given by an equation $ax + by = c$, so we can think of

$$\{(a,b,c) \mid a,b,c \in \mathbb{R}\}.$$

BUT this misses when two equations represent the same line

e.g. $(x + y = 1) \simeq (2x + 2y = 2)$

To make things simpler, let's just consider lines through (0,0):



$$L: \quad ax + by = 0$$

A line through the origin is uniquely specified by

A moduli space of lines through the origin is

4 More generally,

$$\left\{\begin{matrix}\text{lines through}\\ \vec{0} \text{ in } \mathbb{R}^n\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{unit vector}\\ \text{up to } \pm 1\end{matrix}\right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{points in} \\ \mathbb{P}^{n-1} \end{array} \right\}$$

where $\mathbb{P}^{n-1} = S^{n-1} / \text{identify antipodes}$

Most generally,

$$\left\{ \begin{array}{c} \text{k-dim'l subspaces} \\ \text{in } \mathbb{R}^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{k-frame} \\ \text{up to} \\ GL_k(\mathbb{R}) \end{array} \right\}$$

$$\updownarrow$$

$$\left\{ \begin{array}{c} \text{points in} \\ Gr(k,n) \end{array} \right\}$$

where $Gr(k,n)$ = Grassmannian manifold.

Here are same more complicated ones:

5 Complex/hyperbolic surfaces

$\Big\{$ complex/hyperbolic structures on

g holes $\Big\}$ $\longleftrightarrow$ $\Big\{$ Fenchel-Nielsen coordinates $\Big\}$

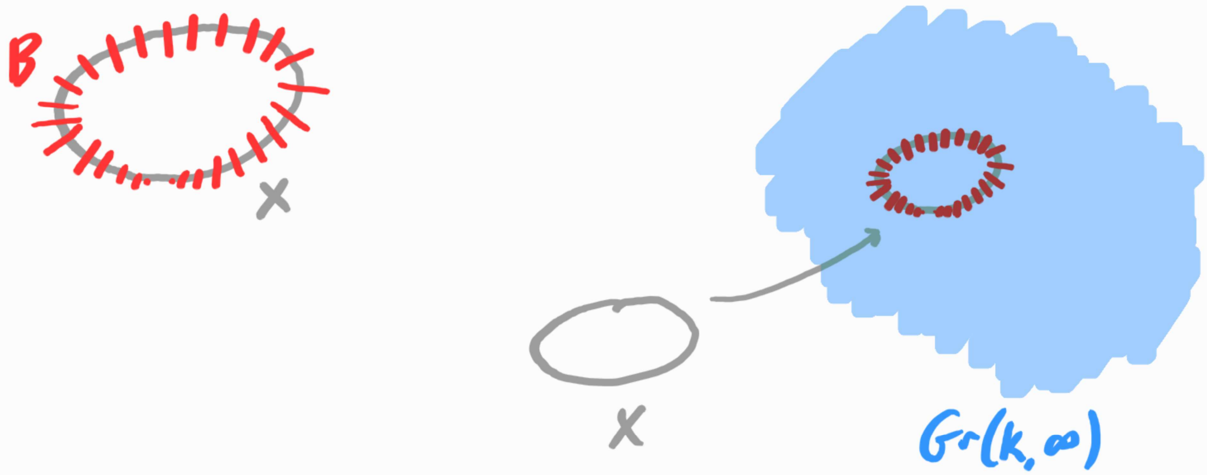
$\updownarrow$

$$\mathbb{R}^{6g-6}$$

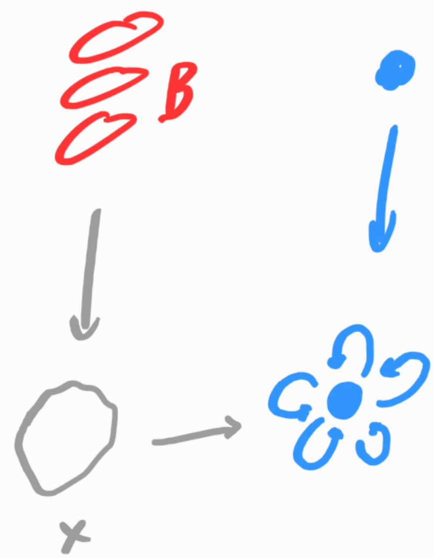
6 Vector bundles and G-bundles

(Rank k vector)

$$\left\{\begin{array}{c}\text{bundles on } X \\ \text{up to iso.}\end{array}\right\} \longleftrightarrow Gr(k,\infty)$$



$$\left\{\begin{array}{c}\text{principal} \\ G\text{-bundles} \\ \text{up to iso.}\end{array}\right\} \longleftrightarrow BG$$



$$\boxed{\text{Lines on Cubic Surfaces}}$$