PRETALK

I. Serre's ramification theory

II. Geometric ramification theory : $\mathbb{Z}/p$-covers of curves

III. ~~Stacky curves?~~ ASW extensions : $\mathbb{Z}/p^2$-covers?

I. <u>Def</u>: A <u>local field</u> is a <u>completed discretely valued field</u> K with finite residue field $k = \mathcal{O}_K/\mathfrak{m}_K$ where $\mathcal{O}_K = \{v(x) \geq 0\}$
$\mathfrak{m}_K = \{v(x) > 0\}$.

Ex: $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is a dvr with $\mathbb{Q}_p = \mathrm{Frac}\,\mathbb{Z}_p$ a local field <u>of char. 0</u>.

Ex: $\mathbb{F}_p((t))$ = Laurent series over $\mathbb{F}_p$ is a local field <u>of char. p</u>.

<u>Theorem</u>: Every local field is a finite ext. of one of these.

I'm mostly interested in $L/\mathbb{F}_p((t))$, the "equicharacteristic case".

Fix a Galois ext. L/K with gp. $G \cong \mathbb{Z}/p\mathbb{Z}$.

<u>Theorem</u> (Serre) : Such ext's are classified up to isomorphism by their <u>ramification jump</u>:

$m = -v_K(\alpha)$ where α is a certain elt. "generating" L/K.

Fact: cyclic p-ext's (in char. p) are all of the form $L = K(\wp^{-1}(\alpha)) := K[y]/(y^p - y - \alpha)$
for some $\alpha \in K$, $\alpha \neq \beta^p - \beta$.

Then the ram. jump is $m = -v_K(\alpha)$ for this α (AS generator).

Ex: $\begin{array}{c} L \\ | \\ \mathbb{F}_p((t)) \end{array}$: $y^p - y = t^{-m} \rightsquigarrow v(t^{-m}) = -m \rightsquigarrow$ jump $= m$.

<u>Interpretation w/ higher ramification gps</u>

<u>Prop</u>: The ramification jump m is the break in the ramification filtration

$$G \supseteq G_0 \supseteq G_1 \supseteq \ldots$$

where $G_s = \{\sigma \in G \mid v_L(\sigma(x) - x) \geq s+1 \ \forall x \in \mathcal{O}_L\}$

$\boxed{\text{II.}}$ Let $\varphi: Y \longrightarrow X$ be a cover of curves over k (char. p), Galois, w/ gp. $G (= \mathbb{Z}/p\mathbb{Z})$.

Birationally,

$$\begin{array}{c} Y \\ \downarrow \\ X \end{array} \quad \longleftrightarrow \quad \begin{array}{ccc} k(Y) = L & \overset{AS}{=} & K(\wp^{-1}(f)) \text{ for } f \in K \\ | & & | \\ k(X) = K & & \end{array}$$

A rational function $f: X \dashrightarrow \mathbb{A}^1_k$ can be viewed as a <u>regular</u> function $X \to \mathbb{P}^1$ (poles of $f \longmapsto \infty$).

We will see that $\mathrm{ord}(f) =$ ramification jump

Ex:
$$\begin{array}{ccccc} & Y & \longrightarrow & \mathbb{P}^1 & [u:v] \\ & \downarrow & & \downarrow & \rotatebox{90}{$\mapsto$} \\ y^p - y = t^{-m} & \mathbb{A}^1 & \xrightarrow{f} & \mathbb{P}^1 & [u^p : v^p - u^{p-1}v] \end{array}$$