A stacky compactification of the ring of Witt vectors

INTRO

Recall from Galois theory that a cyclic Galois extension of degree n is of the form

$$L = k[x]/(x^n - \alpha) \quad \text{for some } \alpha \in k^*$$

when k contains all nth roots of unity, and something similar in general.

__Kummer theory__: $\{\text{cyclic field ext. } L/k\} \xleftrightarrow{1:1} k^\times/k^{\times n}$ (when $\zeta_n \in k$)

$$L = k(\sqrt[n]{\alpha}) \longleftarrow \alpha.$$

However, this fails in characteristic $p > 0$ (1 is the only pth root of unity)

__Artin–Schreier theory__ (for order p extensions; Artin-Schreier-Witt for p^n) classifies cyclic extensions in char. p:

- every Galois ext. L/k of order p is of the form
$$L = k[x]/(x^p - x - \alpha) \quad \text{for some } \alpha \in k$$
(if $G = \langle \sigma \rangle \cong \mathbb{Z}/p$, then $\sigma \cdot x = x + 1$)
- there is a correspondence $\{\text{cyclic } p\text{-ext. } L/k\} \xleftrightarrow{1:1} k/\wp(k)$ where $\wp(k) := \{\beta^p - \beta \mid \beta \in k\}$
$$L = k(\wp^{-1}(\alpha)) \longleftarrow \alpha.$$

__Theorem__ (Serre, Local Fields, Ch. IV): Cyclic p-extensions of a __local__ field k with alg. closed residue field of char. p are classified up to isomorphism by their __ramification jump__:

$$m = -v_k(\alpha) \text{ is the jump of } L/k \text{ where } L = k(\wp^{-1}(\alpha)).$$

Takeaway: classifying cyclic structures in char. p requires finer arithmetic invariants, like the ramification jump, than in char. 0.

ARTIN–SCHREIER–WITT THEORY

References: Witt's original article (German); Lang's Algebra, exercises on p. 330
recommend ☺

Idea: replace $k/\wp(k)$ with something more general

Idea 2: write cyclic p^n-ext. as a tower of p-extensions ...

Ex: A Galois ext. L/k w/ group $\mathbb{Z}/p^2\mathbb{Z}$ can be written

$$\begin{array}{l} L = M[z]/(z^p - z - \beta) = k[x,z]/\underbrace{(f_1, f_2)}_{\text{example of a Witt vector of length 2}} \\ \quad | \\ M = k[x]/(x^p - x - \alpha) \\ \quad | \\ k \end{array}$$

(in contrast, a $\mathbb{Z}/p \times \mathbb{Z}/p$ ext. can be written as a tower in multiple ways)

The ring of Witt vectors $\mathbb{W}(k)$ is a construction that allows us to "lift" certain structures over k (char. p) to a ring $\mathbb{W}(k)$ of char. 0.

Ex. of structures we might lift:
- $\mathbb{W}(\mathbb{F}_p) = \mathbb{Z}_p$ (the p-adic integers)
- the Frobenius $F: x \longmapsto x^p$ lifts to a ring map $\mathbf{F}: \mathbb{W}(k) \longrightarrow \mathbb{W}(k)$
- viewing elts of $\mathbb{Z}_p$ as "p-adic power series" $\sum a_i p^i$, the addition and multiplication of the coeff's do not extend to $\mathbb{Z}_p$; rather, one has to "carry remainders"

  e.g. $(\ldots + (p-1)p^i + \ldots) + (\ldots + p^i + \ldots) = (\ldots + 0p^i + (a_{i+1}+1)p^{i+1} + \ldots)$

So what is $\mathbb{W}(k)$? As a set, $\mathbb{W}(k) = \{$ sequences $(a_0, a_1, \ldots)$ in $k\}$.

To accommodate/generalize the power series operations above, one defines $+$ and $\cdot$ in $\mathbb{W}(k)$ by: $(a_0, a_1, \ldots) + (b_0, b_1, \ldots) := (S_0(a_0, b_0), S_1(a_0, a_1, b_0, b_1), \ldots)$

and $(a_0, a_1, \ldots) \cdot (b_0, b_1, \ldots) := (P_0(a_0, a_1), P_1(a_0, a_1, b_0, b_1), \ldots)$

where S_i, P_i are certain symmetric, integral polynomials (which are complicated)

(always have $S_i = X_i + Y_i + \ldots$) e.g $S_0 = X_0 + Y_0$, $S_1 = X_1 + Y_1 + \dfrac{X_0^p + Y_0^p - (X_0+Y_0)^p}{p}$, etc.

Facts: (1) $\mathbb{W}(k)$ is a ring of char. 0

(2) $\mathbb{W}(\mathbb{F}_p) \cong \mathbb{Z}_p$ as rings (these operations capture $+, \cdot$ for p-adic power series)

(3) The Frobenius $F: k \to k$ extends to a ring map

$$F: \mathbb{W}(k) \to \mathbb{W}(k), \quad (a_0, a_1, \dots) \mapsto (a_0^p, a_1^p, \dots)$$

(4) $k \longmapsto \mathbb{W}(k)$ extends to a <u>functor</u>

$$\mathbb{W}: Alg_k \longrightarrow CommRing$$
$$A \longmapsto \mathbb{W}(A)$$

(5) Therefore, we have defined an <u>affine</u> <u>k-scheme</u> $\mathbb{W}$.

(6) $\mathbb{W} \cong \varprojlim \mathbb{W}_n$ where $\mathbb{W}_n = \{\text{length } n \text{ sequences } (a_0, a_1, \dots, a_{n-1})\}$

(7) <u>Theorem</u>: Let char $k = p$ and $n \geq 1$. Then

$$\{\text{cyclic } p^n\text{-ext. } L/k\} \xleftrightarrow{1:1} \mathbb{W}_n(k)/\wp(\mathbb{W}_n(k))$$

where $\wp = F - 1$

$$L = k(\wp^{-1}(\underline{\alpha})) \longleftarrow \underline{\alpha}.$$

GARUTI'S COMPACTIFICATION

Motivation: Let $\varphi: Y \to X$ be a cover of curves over k (char. p), and suppose φ is Galois with group G ($\cong \mathbb{Z}/p^n\mathbb{Z}$)

$f: X \dashrightarrow \mathbb{W}_n$

Birationally, we can study φ by looking at the ext. of function fields

$$k(Y)/k(X) =: L/K \overset{AS(W)}{\Longrightarrow} L = K(\wp^{-1}(f)) \text{ for some}$$

rat'l witt-vector-valued function f on X,

But this misses important data at a finite subset of points of X.

Garuti's idea: just like rat'l functions $X \dashrightarrow \mathbb{A}^1$ can be viewed as regular functions $X \to \mathbb{P}^1$, there is a (projective) compactification $\overline{\mathbb{W}_n} \supseteq \mathbb{W}_n$ s.t. rat'l functions $X \dashrightarrow \mathbb{W}_n$ correspond to regular maps $X \to \overline{\mathbb{W}_n}$.

Def (Garuti): Set $n \geq 1$ and define ~~locally rigid spaces~~ (schemes)

$$(\overline{\mathbb{W}}_1, \mathcal{O}_{\overline{\mathbb{W}}_1}(1)) := (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$$

and $$(\overline{\mathbb{W}}_n, \mathcal{O}_{\overline{\mathbb{W}}_n}(1)) := (\mathbb{P}(\mathcal{O}_{\overline{\mathbb{W}}_{n-1}} \oplus \mathcal{O}_{\overline{\mathbb{W}}_{n-1}}(p)), \mathcal{O}_{\mathbb{P}}(1)) \text{ for } n \geq 2,$$

where $\mathbb{P}(E)$ is the projective bundle of an algebraic vector bundle E and $\mathcal{O}_{\mathbb{P}}(1)$ is the corresponding tautological bundle.

Main features for us: (1) Tower of $\mathbb{P}^1$-bundles $\cdots \to \overline{\mathbb{W}}_3 \to \overline{\mathbb{W}}_2 \to \overline{\mathbb{W}}_1 = \mathbb{P}^1$
(a Bott tower)

(2) $\overline{\mathbb{W}}_n$ is a projective scheme (even toric variety)

(3) Open immersions $j_n : \mathbb{W}_n \hookrightarrow \overline{\mathbb{W}}_n$

(4) $\mathbb{W}_n \curvearrowright \mathbb{W}_n$ with vector translation extends to $\mathbb{W}_n \curvearrowright \overline{\mathbb{W}}_n$

Garuti: "equivariant compactification"

(5) Ramification data (generalization of ramification jump in $n=1$ case) of a cover of curves $Y \to X$ can be determined explicitly by:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\mathbb{Z}/p^n"'] & \overline{\mathbb{W}}_n \arrow[d, "\wp = F-1 \text{ extends to } \overline{\mathbb{W}}_n"] \\ X \arrow[r, "f"] & \overline{\mathbb{W}}_n \end{tikzcd}$$

this tells us about ramification data $\quad f^*B_n \rightsquigarrow B_n := \overline{\mathbb{W}}_n \setminus j_n(\mathbb{W}_n)$ boundary divisor

Ex:
$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\mathbb{Z}/p"'] & \mathbb{P}^1 \arrow[d] & {[u:v]} \arrow[d, mapsto] \\ X \arrow[r, "f"] & \mathbb{P}^1 & {[u^p; v^p - u^{p-1}v]} \end{tikzcd}$$

$Y: y^p - y = t^{-m} \rightsquigarrow$ ramification jump $= m$
$\downarrow$
$X = \mathbb{A}^1$, uniformizer t at P

where f is a rat'l function s.t. $k(Y)/k(X) \cong k(t)[y]/(y^p - y - f(x))$.
$X \overset{f}{\dashrightarrow} \mathbb{A}^1$

$\boxed{\text{Here, order of the pole of } f = \text{ram. jump.}}$

Explicitly, the ramification divisor $R = \sum_{P \in X} e_P P$ of $Y \to X$ determines a map $X \xrightarrow{f} \overline{W}_n$ such that $\mathrm{ord}_P(f^* B_n)$ = top ramification jump in higher ramification filtration of inertia gp. at P;

or, $\deg(f^*\mathcal{O}_{\overline{W}_n}(1)) = \underbrace{\mathrm{cond}(Y/X, P)}_{\text{conductor of the extension at } P} - 1$

STACKY CURVES

Working definition: a <u>stack</u> is a scheme X (or variety if you prefer) with extra "data" attached along divisors $D \subset X$. The original X is called the <u>coarse space</u>. Write $\mathcal{X}$.

When X is a curve, this means a stack over X, written $\mathcal{X} \xrightarrow[\text{data}]{\text{forget}} X$, and called a <u>stacky curve</u>, consists of a choice of "data" at each point $x \in X$.

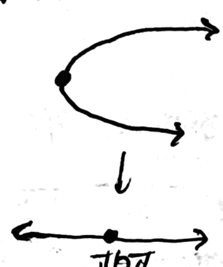
The "data" we attach to $x \in X$ is an abstract "group of automorphisms".

(these objects are used to represent <u>moduli problems</u>, which often parametrize objects having nontrivial automorphisms)

Ex: Let $Y \to X$ be a cyclic cover of curves, say given by the equation

$y^n = f(x)$ (f is a rat'l function on X; branch pts. = zeroes/poles of f)

Then the quotient Y/G, $G = \mathrm{Gal}(Y/X) \cong \mathbb{Z}/n\mathbb{Z}$, should really be viewed as a stacky curve, e.g.



$Y : y^2 = x$

$\downarrow$

$X = \mathbb{P}^1, \mathbb{A}^1$, etc.

Write $[Y/G]$ for the stack.

In general, every <u>tame</u> stacky curve is locally of <u>this form</u>.

char. $p \nmid$ order of any automorphism group

$[Y/G]$ where $Y : y^n = f(x)$, $G \cong \mathbb{Z}/n\mathbb{Z}$

A <u>wild</u> stacky curve has automorphism gps. of order divisible by p, and here's how to handle them.

Locally, they should be modelled by quotients of the form $[Y/G]$, where

$$G = \mathbb{Z}/p \quad \begin{array}{c} Y : y^p - y = f(x) \\ \downarrow \\ X \end{array} \qquad f \text{ a rat'l function on } X$$

To introduce this structure arbitrarily (along a divisor $D \subset X$), I introduced the <u>Artin–Schreier root stack</u> construction, which is defined as a pullback:

$$\begin{tikzcd} \mathcal{X} = \wp_m^{-1}(D/X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[\mathbb{P}(1,m)/\mathbb{G}_a]} \arrow[d, "\Psi"] \\ X \arrow[r, "D"] & {[\mathbb{P}(1,m)/\mathbb{G}_a] = [\mathrm{Th}_k(m)/\mathbb{W}_1]} \end{tikzcd}$$

ext. of $x^p - x$ to the compactification

stack with $\mathbb{Z}/p$-automorphism gp. attached along D

ram. jump

need a little more data than just D

Here, $\mathbb{P}(1,m)$ is the <u>weighted projective line</u>:

$$\overset{\mathbb{Z}/m\mathbb{Z}}{\underset{\infty}{\longleftarrow\!\!\bullet\!\!\longrightarrow}} \quad \mathbb{P}^1$$

coarse space

(Alternatively, $\mathbb{P}(1,m) = [\mathbb{A}^2 \setminus \{0\} / \mathbb{G}_m]$ where $\mathbb{G}_m$ acts with weights $1, m$.)

Fact: $\mathbb{P}(1,m)$ is a classifying space for triples (L, s, f) where $L \to X$ is a line bundle and $s \in H^0(X, L)$, $f \in H^0(X, L^{\otimes m})$ are non-simultaneously vanishing sections. So <u>this</u> is what you're really pulling back along.

Goal: generalize this to an "Artin-Schreier-Witt root stack", which would allow one to introduce automorphism gps. like $\mathbb{Z}/p^n\mathbb{Z}$ along divisors of a scheme.

To begin, view $\mathbb{P}(1,m)$ as a (tame) root stack (this is how stack structure is introduced away from the wild case)

There should be a suitable generalization of the universal stack $[\mathbb{P}(1,m)/\mathbb{G}_a]$, say $[\underbrace{\overline{\mathbb{W}_n}(m_1,\ldots,m_n)}_{\text{some "stacky modification" of Garuti's } \overline{\mathbb{W}_n}}/\mathbb{W}_n]$.

<u>Def</u>: The <u>compactified Witt stack</u> for a sequence of positive integers $(m_1,\ldots,m_n)$ is defined inductively to be a root stack (of order m_n):

$$\overline{\mathbb{W}_n}(m_1,\ldots,m_n) \xrightarrow{m_n\text{-th root}} \overline{\mathbb{W}_n}(m_1,\ldots,m_{n-1},1).$$

The base cases are $\overline{\mathbb{W}_n}(1,\ldots,1) := \overline{\mathbb{W}_n}$ and

$\overline{\mathbb{W}_m}(m_1,1,\ldots,1) =$ pullback of $\mathbb{P}(1,m) := \overline{\mathbb{W}_1}(m)$ in the tower $\longrightarrow \overline{\mathbb{W}_3} \longrightarrow \overline{\mathbb{W}_2} \longrightarrow \overline{\mathbb{W}_1} = \mathbb{P}^1$.

Next steps: study the morphisms $X \longrightarrow [\overline{\mathbb{W}_n}(m_1,\ldots,m_n)/\mathbb{W}_n]$ and use them to define an <u>Artin-Schreier-Witt root stack</u> of X.

In principle, I can then begin to classify stacky curves w/ automorphism groups $\mathbb{Z}/p^n\mathbb{Z}$ — done for $\mathbb{Z}/p\mathbb{Z}$.

Appendix

① $$\left.\begin{array}{c} L \\ | \\ M \\ | \\ k \end{array}\right] \mathbb{Z}/p^2\mathbb{Z}$$

is determined by a pair of AS equations:

$$y^p - y = \alpha \quad \text{and} \quad z^p - z = \frac{\alpha^p + y^p - (\alpha+y)^p}{p} + \beta$$

for some Witt vector (α, β).

② Ghost components: $a = (a_0, a_1, a_2, \dots) \rightsquigarrow a^{(*)} = (a^{(0)}, a^{(1)}, a^{(2)}, \dots)$

where $$a^{(n)} = \sum_{i=0}^{n} p^i a_i^{p^{n-i}}$$

Going backward, a is determined by:

$$a_0 = a^{(0)} \quad \text{and} \quad a_n = \frac{1}{p^n} a^{(n)} - \sum_{i=0}^{n-1} \frac{1}{p^{n-i}} a_i^{p^{n-i}}.$$

Punchline: $(a+b)^{(n)} = a^{(n)} + b^{(n)}$ and $(ab)^{(n)} = a^{(n)} b^{(n)}$ in $W(k)$.

③ Generating function interpretation:

$$f_a(t) := \prod_{n=1}^{\infty} (1 - a_n t^n) \rightsquigarrow -t \frac{d}{dt} \log(f_a(t)) = \sum_{n=1}^{\infty} a^{(n)} t^n.$$

Punchline: $f_a(t) f_b(t) = f_{a+b}(t)$.

④ $W(A)$ is a ring with identity $(0,0,0,\dots)$ and unit $(1,0,0,\dots)$.